

Comparison of different numerical methods for calculating stress intensity factors in analysis of fractured structures

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ARTICLE INFO

Article history:

Received:

December 2017.

Revised:

June 2018.

Accepted:

September 2018.

Keywords:

Stress intensity factors

Fracture mechanics

Adaptive GFVM

Adaptive GFEM

XFEM

Abstract:

In this research, an efficient Galerkin Finite Volume Method (GFVM) along with the h -refinement adaptive process and post-processing error estimation analysis is presented for fracture analysis. The adaptive strategy is used to produce more accurate solution with the least computational cost. To investigate the accuracy and efficiency of the developed model, the GFVM is compared with two versions of the Finite Element Method known in solid mechanics, the adaptive Galerkin Finite Element Method (GFEM) and Extended Finite Element Method (XFEM), for the two dimensional fracture analysis of structures. After the discretization of the governing equations, the above three methods are implemented in FORTRAN. In the adaptive GFVM and GFEM methods, the discrete crack concept is used to model the crack surface, but in the XFEM, the crack surface is modeled through the enrichment of the displacement approximation around the crack. Several test cases are used to validate the developed dimensional numerical models for the analysis of cracked structures. After verification, the fracture analysis of a plate under pure mode I and mixed mode I/II is performed using the above-mentioned numerical methods. The numerical results show that three methods accurately calculate the stress intensity factors. The average percent error of the XFEM, adaptive GFEM and adaptive GFVM is 0.88%, 2% and 1.75%, respectively. The results show that the CPU time of the adaptive GFVM is 5.5 and 3 times less than the XFEM and adaptive GFEM, respectively.

1. Introduction

The high accuracy and low computational cost of numerical methods in comparison with the experimental and analytical methods cause more researchers to focus on these methods.

The limitations of the experimental and analytical methods in realistic and complex problems, especially in fracture mechanics, have resulted in rapid progress of the numerical methods in fracture mechanics problems. Obviously, each class of numerical methods is developed for special problems; as a result, their accuracy varies in different problems.

The Finite Difference Method discretizes the computational domain by a finite number of segments. The Finite Difference Method suffers from numerical errors and inefficiencies to solve the boundary value problems on irregular computational domains. As a result, the analysis of complex geometries in multiple dimensions is difficult using the Finite Difference Method. This issue motivates researchers to use the integral form of the governing equations such as the Finite Element Method and Finite Volume Method (Yip 2005[23]).

The Finite Element Method is a well-known numerical method for structural analysis, and problems in solid mechanics are often addressed by this method due to the excellent numerical results. This method can be easily extended to higher order shape functions, but this process produces large block-matrices (Jasak and Weller 2000[7]).

One can either use the adaptive methods or the enrichment functions to enrich the nodes of the elements cut by discontinuity to obtain high accuracy numerical results. The adaptive Finite Element Method is simple and automatic, so it can be applied to any engineering problems

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with complex geometry. The appropriate size of elements is predicted using an error estimation analysis and a new mesh is created using the adaptive algorithms (Vasiliauskiene *et al* 2006[21]). Khoei *et al.*, used the Zienkiewicz–Zhu error estimator and the super-convergent patch recovery technique to crack propagation analysis using the Finite Element Method. (Khoei *et al* 2008[9]). Limtrakarn *et al.*, used the adaptive Finite Element Method and photo elastic technique to calculate the stress intensity factors, K_I and K_{II} . They used the eight-node quadrilateral elements around the crack tip. In the adaptive Finite Element Method, small elements are used around the crack to show high stress gradient at the crack tip, while coarse elements are used away from the discontinuity to reduce the total number of elements and, consequently, to reduce the computational time (Limtrakarn *et al* 2010[12]). Murotani *et al.*, used a hierarchical mesh to increase the speed of the classic adaptive methods in fracture analysis. They showed that the speed of computations can be increased using the hierarchical mesh compared to the Delaunay triangulation method (Murotani *et al* 2013[17]).

Meng and Wang used the XFEM on structured square mesh for crack propagation modeling in power-law creep materials. The maximum principal stress criterion has been used to determine the direction of crack propagation. They predicted the path of creep crack propagation in a plate with an edge crack, notch bar specimen and compact tension specimen (Meng and Wang 2014[15]). The combination of the XFEM and an analytical technique has been used to determine the strain energy release rates. In fact, the stiffness matrix derivative has been calculated using the crack-tip functions of the XFEM (Waisman 2016[22]). Jiang *et al.*, performed the fracture analysis of composite materials using a combination of the XFEM and cohesive element technique. The cohesive elements have been used to simulate delamination of composite materials (Jiang *et al* 2016[8]).

The Finite Volume Method is a second–order accurate numerical approximation which is based on the integral form of the governing equations. This numerical method uses a segregated solution procedure and the nonlinearity are considered through an iterative process (Jasak and Weller 2000[7]). The first attempt was made by Demirdzic and Muzaferija to use the Finite Volume Method in solid mechanics. They discretized the equilibrium equation written in an integral form using the Finite Volume Method and performed stress analysis on unstructured mesh (Demirdzic and Muzaferija 1994[3]). Bailey and Cross used the Finite Volume Method for stress analysis of 3D problems on unstructured mesh. The control volume used is generated by connecting the elements' centers to the mid points of their sides (Bailey and Cross 1995[2]).

Ebrahimnejad *et al.*, showed that the accuracy of numerical results can be increased by combining the mesh-less Finite Volume Method and adaptive technique. The

Zienkiewicz-Zhu error estimator and T-Belytschko stress recovery scheme have been used to identify large error regions. In the proposed method, the optimal location of the new nodes are found using genetic algorithm by minimizing the global error (Ebrahimnejad *et al* 2015[5]).

The mesh adaptation method has been used to model the turbulent flow around airfoils using the Finite Volume Method. A data structure for mesh adaptation on unstructured mesh has been presented (Hay and Visonneau 2007[6]). The adaptive finite volume technique has been used to solve second-order partial differential equations using graph-based adaptive mesh refinement method, which requires low computational cost and computational storage (Oliveira and Oliveira Chagas 2015[18]).

Sabbagh-Yazdi *et al* performed thermal stress analysis of concrete dam using the GFVM on unstructured triangular elements. The variation of mechanical properties is considered corresponding to the degree of concrete hydration and concrete temperature (Sabbagh-Yazdi *et al* 2013[19]).

In this study, the advantages and disadvantages of the well-known numerical methods (the Extended Finite Element method, adaptive Finite Element method and adaptive Finite Volume method) in fracture mechanics have been investigated to allow the researchers to choose the optimal method based on the available facilities and expected goals. In the present research, the h–refinement adaptive process and post–processing error estimation analysis have been used. The Aim of the adaptive strategy is to produce more accurate solution with the least computational cost. At the first stage of the adaptive analysis, the computational domain is discretized into a coarse uniform triangular mesh, using an automatic 2D unstructured mesh generator. Then a rough solution of the problem is found on the initial uniform mesh. In the next step, error estimation is performed all over the mesh to find all regions of high relative error. Consequently, if a mesh refinement in a certain region is necessary, a mesh refinement algorithm is simply applied, based on bisecting the longest edge of the selected elements and inserting extra nodes to the computational domain.

In the section 2, the Finite Element Method and governing equations of solid mechanics are introduced. The Cauchy equation and its discretized form using the GFVM are briefly described in section 3. In section 4, adaptive methods, including the error estimation analysis and data transfer analysis are presented. Consequently, the XFEM is briefly described in section 5. In section 6, the interaction integral is given to calculate the stress intensity factors. Finally, the numerical test cases are provided to compare the efficiency of the above mentioned methods.

2. GFEM

The Finite Element Method is well-known in solid mechanics and is mostly used to solve partial differential

equations. The Finite Element Method transforms the partial differential equations into algebraic equations using a simple approximation of unknown variables (Dhatt *et al* 2012[4]).

2.1 Equilibrium Equation

The total potential energy of a body can be obtained from Eq. 1:

$$\Pi = \underbrace{\sum_2^l \int \epsilon^T D \epsilon t dA}_e - \underbrace{\sum \int \delta^e f_b t dA_e - \int \delta^r f_s t dl - \sum \delta_i^e P_i}_{\text{external work}} \quad (1)$$

Where f_b is the body force, f_s is the surface traction and P_i is the concentrated force.

Finally, the equilibrium equation can be found by minimizing potential energy (Eq. 2):

$$K\delta = F \quad (2)$$

Where K is the stiffness matrix, δ is the displacement vector, and F is the force vector.

The stiffness matrix can be defined by Eq.3:

$$[K] = \iiint_V [B]^T [D] [B] dV \quad (3)$$

Since the triangular constant strain element is used, the above integrand is not function of x, y, z and can be taken out of the integral. Therefore the stiffness matrix for a 3-nodes triangular element are given by:

$$[K] = \begin{bmatrix} [k_{ii}] & [k_{ij}] & [k_{il}] \\ [k_{ji}] & [k_{jj}] & [k_{jl}] \\ [k_{li}] & [k_{lj}] & [k_{ll}] \end{bmatrix} \quad (4)$$

Where the sub-matrices are given by Eq. 5:

$$[k_{mn}] = [B_m]^T [D] [B_n] t A ; \quad m, n = i, j, l \quad (5)$$

Where

$$[B_\Gamma] = \frac{1}{2A} \begin{bmatrix} \beta_\Gamma & 0 \\ 0 & \gamma_\Gamma \\ \gamma_\Gamma & \beta_\Gamma \end{bmatrix} ; \quad \Gamma = i, j, l$$

$$\begin{aligned} \beta_i &= y_j - y_l ; \beta_j = y_l - y_i ; \beta_l = y_i - y_j \\ \gamma_i &= x_l - x_j ; \gamma_j = x_i - x_l ; \gamma_l = x_j - x_i \end{aligned} \quad (6)$$

Here (x_i, y_i) , (x_j, y_j) and (x_l, y_l) are the nodal coordinates of nodes i, j and l (Logan 2012 [13]).

3. GFVM

The Finite Volume Method transforms the partial differential equations into algebraic equations based on equilibrium on the element boundaries. The first attempt to use in fluid mechanics has been made in 1971. Then Demirdzic and Muzaferija used this numerical method on unstructured mesh in solid mechanics (Demirdzic and Muzaferija 1994[3]).

3.1 Equilibrium Equation

The Cauchy equilibrium equation can be written as following (Sabbagh-Yazdi *et al* 2013[19]):

$$\nabla \sigma_{ij} + f = \rho \frac{\partial V}{\partial t} \quad (7)$$

Where ρ is the material density, V is the velocity, f is the body forces, and σ_{ij} is the stress tensors.

Hook's law is used for the stress-strain field as Eq. 8:

$$\sigma = D \epsilon \quad (8)$$

The elasticity matrix $[D]$ for plane stress and plane strain condition is given as Eq.9 and 10, respectively:

$$D = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \quad (9)$$

$$D = \frac{E}{(1-2\nu)(1+\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \quad (10)$$

The discretization form of stress tensor is expressed as Eq. 11:

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{1}{A_n} \sum_{l=1}^3 [D_{11} \bar{u}_l \Delta y - D_{12} \bar{u}_l \Delta x]_l \\ \frac{1}{A_n} \sum_{l=1}^3 [D_{12} \bar{u}_l \Delta y - D_{11} \bar{u}_l \Delta x]_l \\ \frac{1}{A_n} \sum_{l=1}^3 [D_{22} \bar{u}_l \Delta y - D_{22} \bar{u}_l \Delta x]_l \end{pmatrix} \quad (11)$$

Here A_n denotes the area of the desired triangular element.

3.2 Discretization of the Governing Equation

The GFVM discretize the Cauchy governing equation using Galerkin theory and integration on the computational domain. The weight function of the Finite Volume Methods (ϕ) is the same as the shape function of the Finite Element Method. The Cauchy equation is multiplied by the weight function, then the integral of the resulting equation over sub-domain is calculated and finally, the discretized form of the Cauchy equation in the direction i is given as the following equation:

$$\begin{aligned} (u_i)_n^{m+1} &= ((\Delta t)_n^m)^2 \left[\frac{f}{\rho} + \frac{3}{2\rho\Omega_n} \left\{ \sum_{l=1}^N (\bar{F}_i^S \cdot \Delta L_i)_l^m \right\} \right] \\ &+ 2(u_i)_n^m - (u_i)_n^m, \quad (i=1,2) \end{aligned} \quad (12)$$

Where $(u_i)_n^{m+1}$ is the displacement of node "n" in the direction "i" at iteration $m+1$. Ω_n is the area of the control volume, N is the number of boundary edges of the control volume Ω (Amiri 2015[1]).

4. Adaptive Methods

Creating the best mesh for a specific problem is the most important objective of adaptive methods. The optimal mesh is one that uses minimal number of nodes which minimizes error function (McNeice and Marcal 1973[14]). First, the error function is calculated on the computational domain, then the mesh should be modified until the estimated error is less than target error. In general, there are h -refinement, p -refinement and hp -refinement techniques to optimize the mesh. The first is used in the present study.

4.1 Error Estimation Analysis

The posteriori error estimator is used to examine the accuracy of numerical solution, among which the post-processing methods estimate the error more accurately than the residual methods (Zienkiewicz *et al* 1999[24]).

The error is the difference between the exact solution and numerical solution, but since the exact solution (σ_{ij}^{Ex} , u_i^{Ex}) is un-known, the error is estimated as the difference between the numerical solution and recovered solution in the post-processing method. Therefore, the error for the displacement and stress can be obtained as follows:

$$e_s = \sigma_{ij}^{Ex} - \sigma_{ij} \quad , \quad e_u = u_i^{Ex} - u_i \quad (13)$$

$$e_s \cong \sigma_{ij}^* - \sigma_{ij} \quad , \quad e_u \cong u_i^* - u_i \quad (14)$$

Where u_i^* , u_i are the recovered displacement and numerical displacement, respectively. Here σ_{ij}^* , σ_{ij} denote the recovered stress and numerical stress, respectively.

The new size of the element l_i^* which causes the computational error to be less than the target error, can be expressed as follows (Zienkiewicz and Zhu 1987[25]):

$$(H_{new})_i = \frac{(H_{old})_i}{x_i^{1/\phi}} \quad (15)$$

Where

$$x_i = \frac{\omega^* \|e\|_i}{(\|e\|_i)_{Aim}} \quad (16)$$

Where, $(H_{new})_i$ is the new size of the element l_i^* , $(H_{old})_i$ is the old size of the element l_i , ϕ is the minimum of the two values; the singularity order and order of the shape function of the element. $\|e\|_i$ is the error percentage of the element l_i^* , $(\|e\|_i)_{Aim}$ is the target percentage error and the parameter " ω " for the triangular elements is equal to 1.3 (Amiri 2015 [1]).

4.2 Data transfer operator

All variables must be transferred from the old mesh to the new mesh to accelerate the solution process and trace the history of problem in linear and nonlinear elastic problems. In this research, only the displacement field is transferred from the old mesh to the new one due to less numerical dispersion; and the stress-strain field is calculated by

performing stress-strain analysis on the new mesh using the transferred displacement field (Amiri 2015 [1]).

5. XFEM

The most important disadvantage of the numerical methods is the strong dependence of results to the mesh. So the mesh size effect on the numerical results should be investigated. In addition, re-meshing is one of the main challenges of this method for problems include discontinuity, while in the XFEM, re-meshing is not required for crack propagation. The XFEM uses the enrichment functions around the crack to simulate the crack trajectories (Mohammadi 2008[16]).

The displacement field in the XFEM is approximated as Eq. 17:

$$u(x) = \underbrace{\sum_{i=1}^N N_i(x) u_i}_{\text{classic part}} + \underbrace{\sum_{j=1}^M N_j(x) \Psi(x) \bar{a}_j}_{\text{Enriched part}} \quad (17)$$

Where N is the number of nodal points for each element, M is the number of enriched DOFs, $N_i(x)$ is the shape functions of the classic Finite Element Method, $N_j(x)$ is the shape functions of the XFEM, u_i is the standard displacement vector of node i , $\bar{a}_j$ is the enriched DOF related to the basis $\Psi(x)$ which is obtained from the following equations:

$$\begin{aligned} \Psi_1 &= \sqrt{r} \sin(\theta/2) \\ \Psi_2 &= \sqrt{r} \cos(\theta/2) \\ \Psi_3 &= \sqrt{r} \sin \theta \sin(\theta/2) \\ \Psi_4 &= \sqrt{r} \sin(\theta) \cos(\theta/2) \end{aligned} \quad (18)$$

Where the local coordinates r and θ are measured from the crack tip (Khoei 2015[10]). A discontinuous Heaviside function is used to incorporate the displacement jump across the crack surface.

Finally, the stiffness matrix of the element $\mathcal{E}$'s defined in Eq. 19:

$$K_{ij}^e = \begin{bmatrix} K_{ij}^{uu} & K_{ij}^{ua} & K_{ij}^{ub} \\ K_{ij}^{au} & K_{ij}^{aa} & K_{ij}^{ab} \\ K_{ij}^{bu} & K_{ij}^{ba} & K_{ij}^{bb} \end{bmatrix} \quad (19)$$

Where arrays of the stiffness matrix are obtained from Eq. 20:

$$K_{ij}^{rs} = \int_{\mathcal{E}} (B_i^r)^T D B_j^s d\Omega \quad , \quad (r,s=u,a,b) \quad (20)$$

Where matrix B is the derivation of the shape function which can be found in the following equations:

$$B_i^u = \begin{bmatrix} N_{i,x} & 0 \\ 0 & N_{i,y} \\ N_{i,y} & N_{i,x} \end{bmatrix} \quad (21)$$

$$B_i^a = \begin{bmatrix} (N_i H)_{,x} & 0 \\ 0 & (N_i H)_{,y} \\ (N_i H)_{,y} & (N_i H)_{,x} \end{bmatrix} \quad (22)$$

$$B_i^b = \{B_i^{b1} \cdot B_i^{b2} \cdot B_i^{b3} \cdot B_i^{b4}\}^T \quad (23)$$

$$B_i^a = \begin{bmatrix} (N_i F_a)_{,x} & 0 \\ 0 & (N_i F_a)_{,y} \\ (N_i F_a)_{,y} & (N_i F_a)_{,x} \end{bmatrix}, \alpha = b1, b2, b3, b4 \quad (24)$$

Where B_i^u is the matrix of derivative of the classical shape function, B_i^a is the matrix of derivative of the Heaviside enrichment shape functions, and B_i^b is the matrix of derivative of the crack tip enrichment shape function (Mohammadi 2008[16]).

6. Fracture Mechanics

The stress intensity factors and strain energy release rates are the most important parameters required for numerical simulation in fracture mechanics. In this research, the interaction integral (Eq. 25) is used to calculate the stress intensity factors under pure mode I and mixed mode.

$$\begin{aligned} J^{(Act, Aux)} = & \int_A^* [(\sigma_{ij}^{Act} + \sigma_{ij}^{Aux})(u_i^{Act} + u_i^{Aux})_{,l} - \\ & w_S^{(Act, Aux)} \delta_{lj}] q_j dA + \int_{\Gamma_{cr^-} \cup \Gamma_{cr^+}} [w_S^{(Act, Aux)} \delta_{lj} - \\ & (\sigma_{ij}^{Act} + \sigma_{ij}^{Aux})(u_i^{Act} + u_i^{Aux})_{,l}] q n_j ds \end{aligned} \quad (25)$$

Where

$$M = \frac{2}{E} (K_I^{Act} K_I^{Aux} + K_{II}^{Act} K_{II}^{Aux}) \quad (26)$$

Obviously, the stress intensity factors can be calculated by choosing appropriate auxiliary fields (Song and Paulino 2006[20]).

7. Numerical Results

7.1 Verification

7.1.1 A plate with a double edge crack

Let us take a square plate with a double edge crack under a uniform tension as shown in Fig. 1. The plate dimension and crack length are $a=0.35m$ $b=0.7m$, respectively. The mechanical properties of material can be found in Table 1. The fracture analysis is carried out under plane stress. The far-field tension stress is assumed $\sigma=1 Pa$.

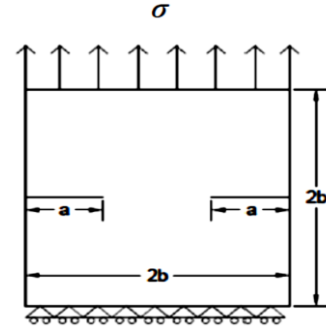


Fig. 1 Geometry and boundary condition for the plate with double edge crack

Table 1 The mechanical properties of material

Mechanical Properties	Values (Unit)
Elasticity Modulus	$E=144.8 GPa$
Poisson's Ratio	$\nu=0.21$
Stress	$\sigma=1 Pa$

Fig. 2 shows the irregular triangular mesh used in fracture analysis using the XFEM. The irregular triangular mesh used by the adaptive methods is presented in Fig. 3. As seen in Fig. 2, the XFEM does not require to model the crack surface geometrically.

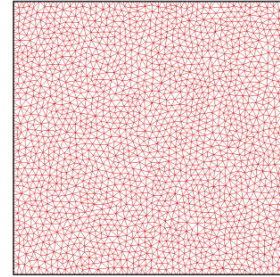


Fig. 2 The unstructured mesh for fracture analysis using XFEM

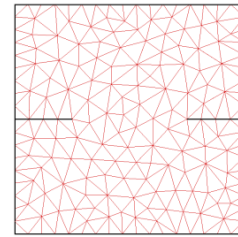


Fig. 3 The unstructured mesh for fracture analysis using the adaptive GFEM and GFVM

A mesh refinement algorithm is applied based on bisecting the longest edge of the selected elements and inserting extra nodes to the computational domain. The sequential refinements are presented in Fig. 4.

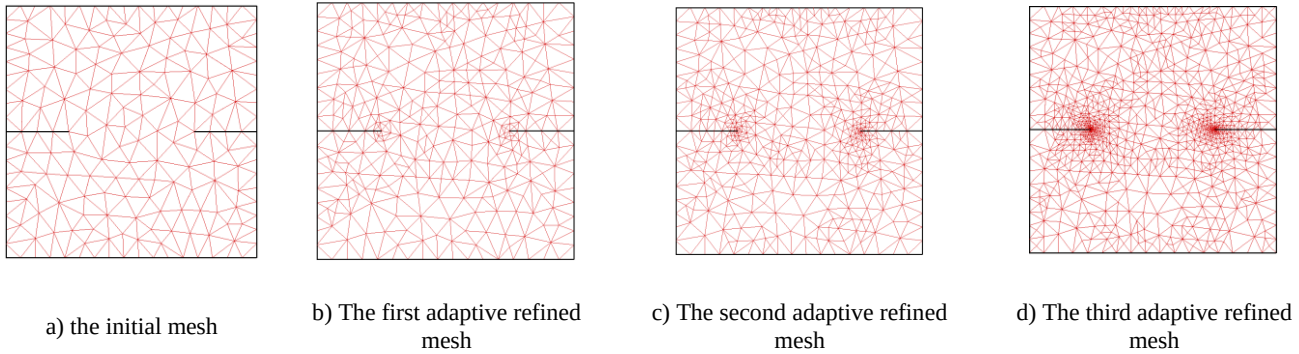


Fig. 4 The refined mesh for a plate with a double edge crack

The optimal mesh uses minimal number of nodes which minimizes error function. This mesh is generated in adaptive mesh strategies using the results of stress analysis, error estimation analysis and refinement process. Therefore, the computational domain is discretized into a coarse triangular mesh in both adaptive methods. Obviously, this coarse mesh does not give acceptable results. But all elements of high relative error are found and refined in adaptive analysis and finally the optimal mesh is achieved considering a target relative error. This optimal mesh is used in fracture analysis using both adaptive methods. On the other hand, mesh does not change during fracture analysis in the XFEM. Therefore, using the coarse grid (such as Fig. 3) will result in unacceptable numerical results. So a fine mesh should be used, especially around the crack. On the other hand, smaller elements are used in the predicted region of crack propagation to accurately predict the crack propagation path in the XFEM.

Since the adaptive GFVM is iterative, the computed displacements of an optional node with coordinates $(0.3319, 0.0542)$ is presented in Fig. 5 to ensure convergence. It can be observed that the displacement of

the numerical solution converge to the exact solution after some iterations.

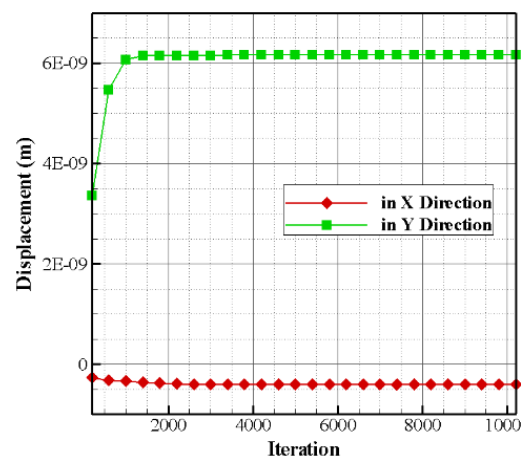


Fig. 5 The convergence of the computed displacement of an optional node in the adaptive GFVM

Fig. 6 illustrates the displacement contour in the direction y'' obtained from the three above-mentioned numerical methods. As expected, all three methods predict the displacement field accurately.

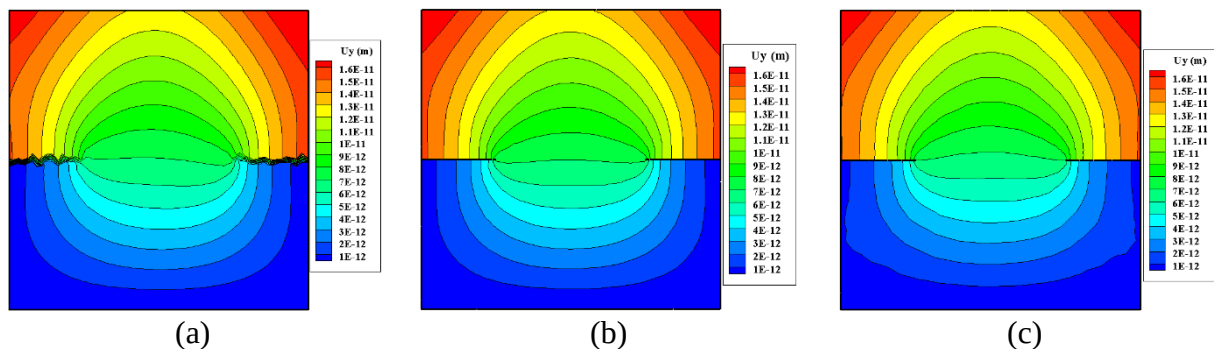


Fig. 6 The displacement contours in normal direction to the crack surface for plate with double edge crack: a) The XFEM b) The adaptive GFEM c) The adaptive GFVM

The enrichment functions are used for enrichment the nodes of the elements cut by discontinuity in the XFEM. The elements cut by the crack are observed in the Fig. 7,

which shows the significant gradients of displacement in normal direction to the crack surface and a clear illustration of the crack.

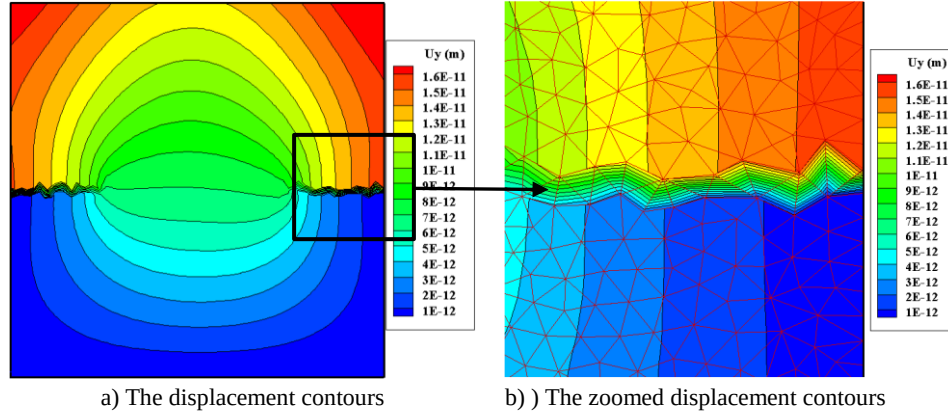


Fig. 7 The displacement contours in normal direction to the crack surface using the XFEM on unstructured mesh

The crack surface is modeled through the enrichment of the displacement approximation around the crack in the XFEM. finally, the crack surface is a completely smooth surface on the structured mesh, but it is not a smooth surface on the unstructured mesh. The crack surface

becomes smoother using the smaller elements. This test case has been performed on both structured and unstructured mesh and the numerical results of the XFEM for structured mesh are presented in Figs. 8 and 9.

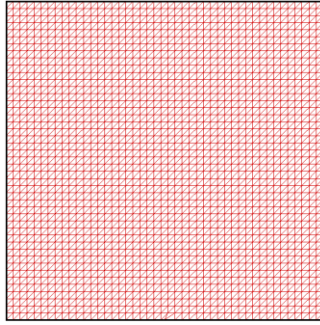


Fig. 8 The structured meshes for fracture analysis using the XFEM

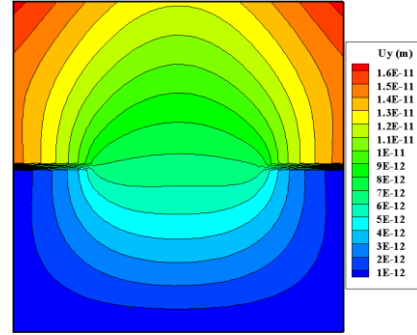


Fig. 9 The displacement contours in normal direction to the crack surface using the XFEM on structured mesh

In order to investigate the effect of radius of interaction integral on SIFs, the values of the SIFs for different radii are plotted in Fig. 10. As it can be seen, the interaction integral is less sensitive to the selected radius.

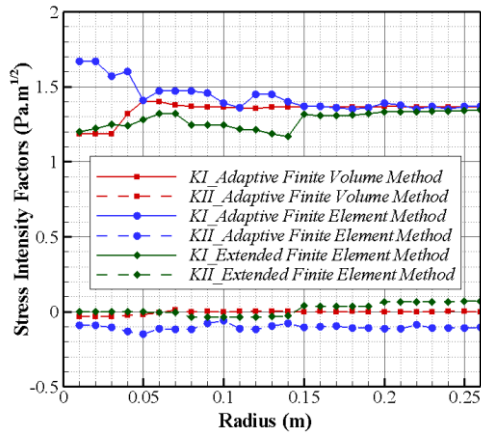


Fig. 10 The effect of radius of interaction integral on SIFS

Eq. 27 is used to calculate the analytical stress intensity factor under pure mode I:

$$K_I = \aleph \sigma \sqrt{\pi a} \quad (27)$$

Here $\aleph$ is taken as equal to "1.274" the ratio of length to width $\frac{a}{b}=0.5$ (Kim 1985[11]). The analytical stress intensity factor is given by (Kim 1985[11]), which is equal "1.336", whereas the three methods show little difference. In fact, the percentage error of K_I is 0.75%, 2.5% and 2.1% in the XFEM, adaptive GFEM and adaptive GFVM, respectively. The CPU time consumption is 3.7, 11 and 15.8 seconds in the adaptive GFVM, adaptive GFEM and XFEM, respectively.

Fig. 11 shows the relative error distribution of the computational domain along the adaptive analysis. Using the estimated percentage error, a new mesh with more uniform error distribution is created.

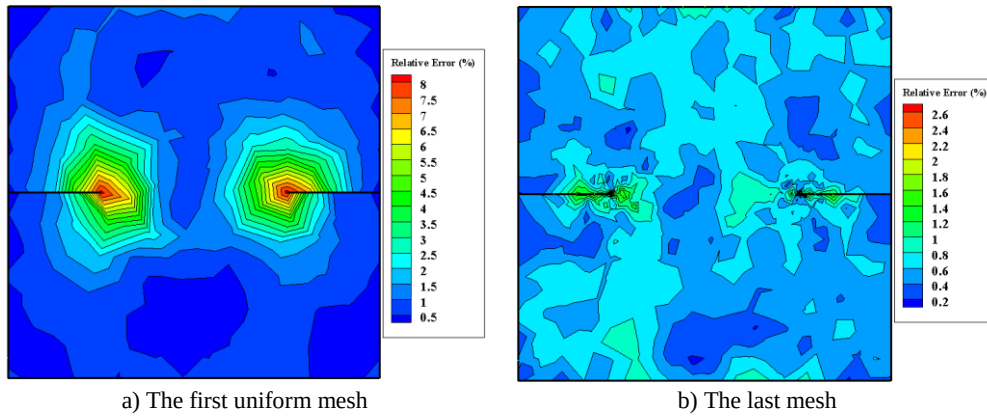


Fig. 11 The relative error distribution for a plate with a double edge crack

The effect of mesh size on the mode I stress intensity factor has been obtained from a sensitivity analysis. Table 2 compares the mode I stress intensity factor, error percentage, CPU time for various meshes shown in Fig.

12. As can be seen in Table 2, increasing the number of nodes drastically increases the computational time.

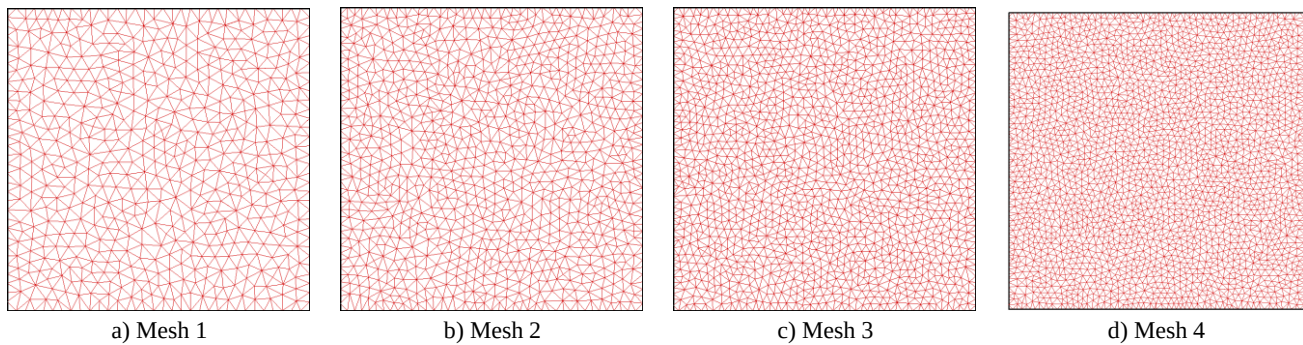


Fig. 12 Different meshes for fracture analysis using the XFEM

Table 2 Comparison of the stress intensity factors for different meshes.

	Analytical	XFEM			
		Mesh 1	Mesh 2	Mesh 3	Mesh 4
Mode I stress intensity factor ($Pa.m^{1/2}$)	1.336	1.25	1.28	1.30	1.334
Error percentage (%)	----	5.86	4.23	1.95	0.15
CPU Time (sec)	----	1.2	4.13	10.07	26.87
Number of nodes	----	632	1021	1578	2477

7.1.2 A plate with a central inclined crack

In this example, a mixed-mode fracture analysis is investigated. A plane stress plate with a central inclined crack is simulated, as depicted in Fig. 13. The geometric parameters, $a=0.07m$ and $b=0.5m$, are used in the fracture analysis. Table 3 presents the mechanical properties of material.

Table 3 The mechanical properties of material

Mechanical Properties	Values (Unit)
Elasticity Modulus	$E=144.8 \text{ GPa}$
Poisson's Ratio	$\nu=0.21$
Stress	$\sigma=1000 \text{ Pa}$

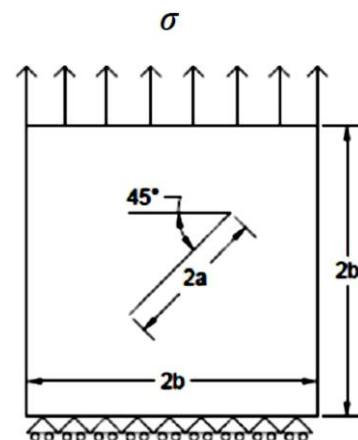


Fig. 13 Geometry and boundary condition for the plate with a central inclined crack

The configuration of the unstructured mesh used by the XFEM is depicted in Fig. 14. The irregular triangular meshes used by the adaptive methods are presented in Fig. 15, which includes the initial and final meshes.

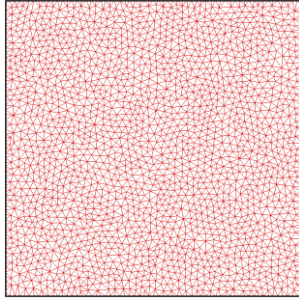


Fig. 14 The unstructured meshes for fracture analysis using XFEM

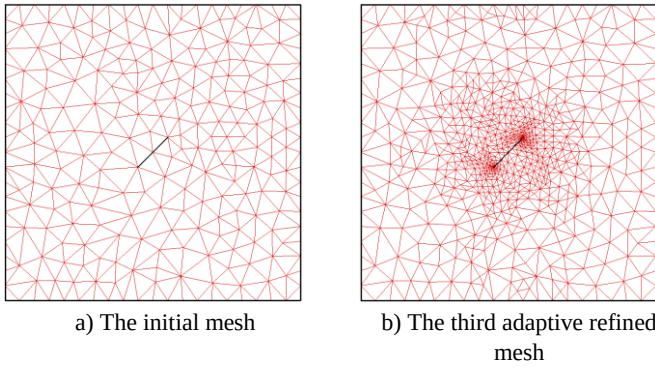


Fig. 15 The unstructured meshes for fracture analysis using the adaptive GFEM and GFVM

The computed displacements of an optional node with coordinates $(0.7594, 0.8821)$ is presented in Fig. 16 to ensure convergence. The root mean square of the displacements (Eq. 28) is presented in Fig. 17. As can be seen, the logarithm of root mean square error (RMSE) of the computed displacements decrease to -14.

$$RMS = \log \sqrt{\frac{\sum_{n=1}^N (u_i^{new} - u_i^{old})^2}{N}} \quad (28)$$

Where N is the total number of nodes, $(u_i^{new})_n$, $(u_i^{old})_n$ are displacement of node n in direction i at the present and previous iteration, respectively.

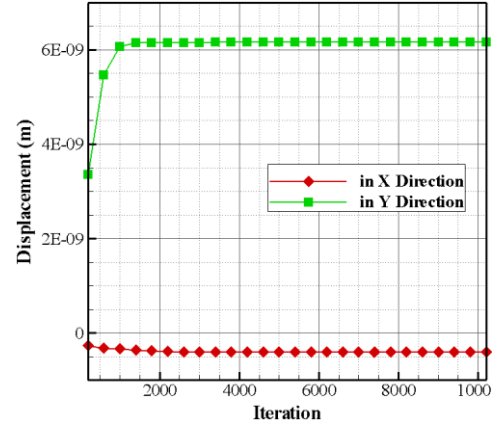


Fig. 16 The convergence of the computed displacement of an optional node in the adaptive GFVM

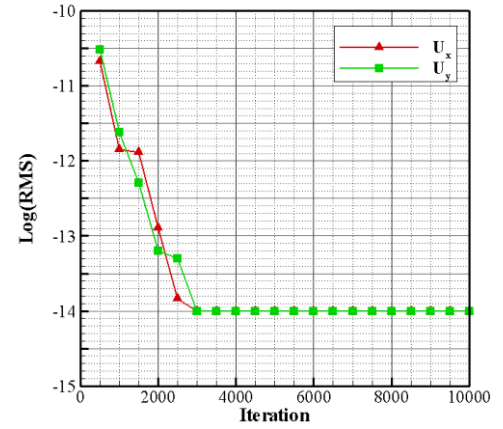


Fig. 17 The convergence history of the logarithm of root mean square error (RMSE)

The y -displacement contours obtained from the three numerical methods have been shown in Fig. 18. Clearly, the discontinuous displacement field across the crack surface is observed. Fig. 19 shows the stress counter in the direction y (σ_{yy}) for the plane stress plate with the inclined central crack using the three numerical methods.

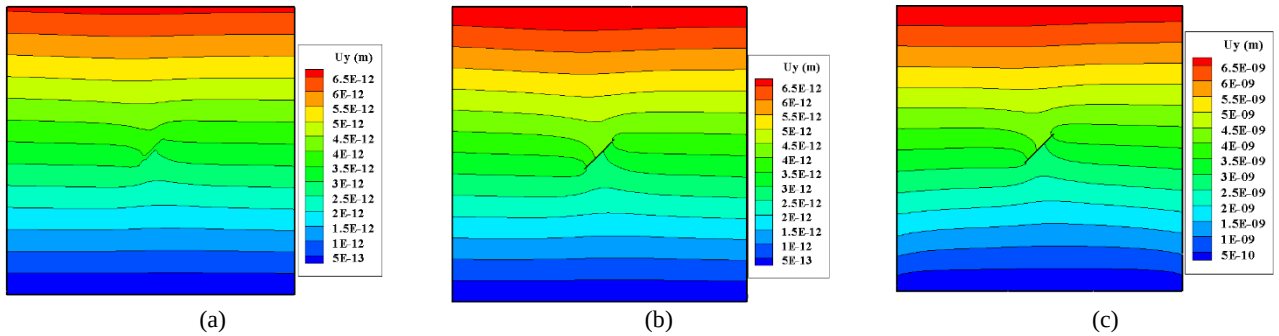


Fig. 18 The displacement contours in normal direction to the crack surface for plate with the inclined central crack: a) The XFEM b) The adaptive GFEM c) The adaptive GFVM

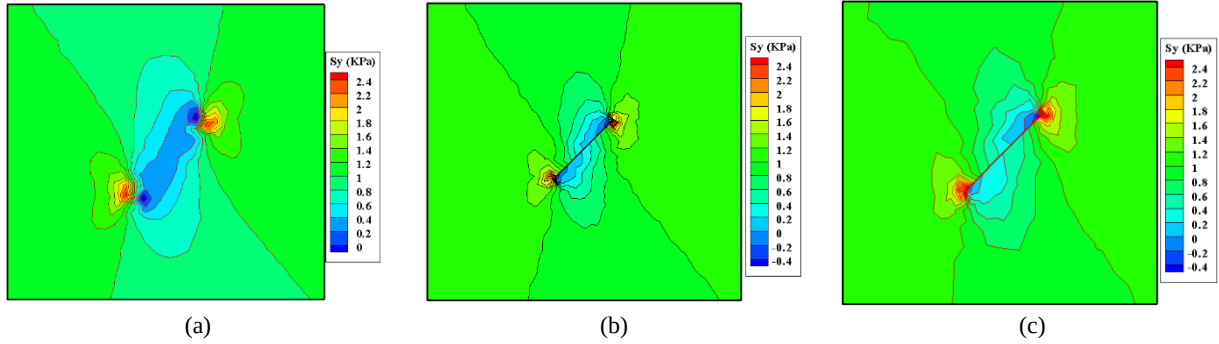


Fig. 19 The stress contours in normal direction to the crack surface for plate with the inclined central crack: a) The XFEM b) The adaptive GFEM c) The adaptive GFVM

Fig. 20 shows the variation of the stress intensity factors of modes 1 and 2 for the different radii of interaction integral, which the patterns seem logical.

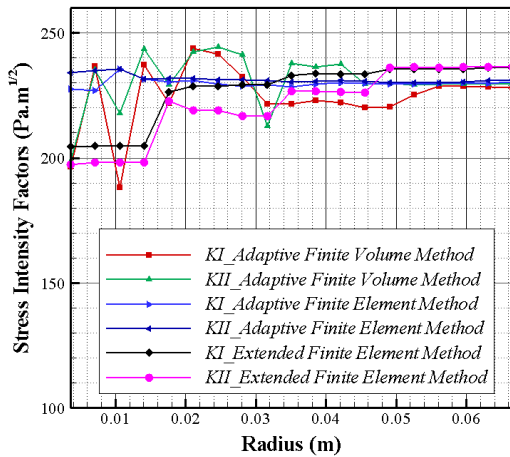


Fig. 20 Variations of the SIFS with respect to different radii of the interaction integral

For this test case, the analytical stress intensity factors are given by the following Equations (Khoei 2015[10]).

$$K_I = \sigma (\cos \beta)^2 \sqrt{\pi a} \quad (29)$$

$$K_{II} = \sigma (\cos \beta) (\sin \beta) \sqrt{\pi a} \quad (30)$$

The mode I and II analytical stress intensity factors are equal to "234" for $\beta=45^\circ$ and the numerical results show good agreement with the available analytical solution. The average percentage error of K_I and K_{II} is 0.97%, 1.43% and 1.86% in the XFEM, adaptive GFEM and adaptive GFVM, respectively. The CPU time consumption is 3, 9 and 22 seconds in the adaptive GFVM, adaptive GFEM and XFEM, respectively.

The percentage relative error of the adaptive analysis on the first and last mesh is presented in Fig. 21. Using the estimated error, a new mesh with more uniform error distribution is created.

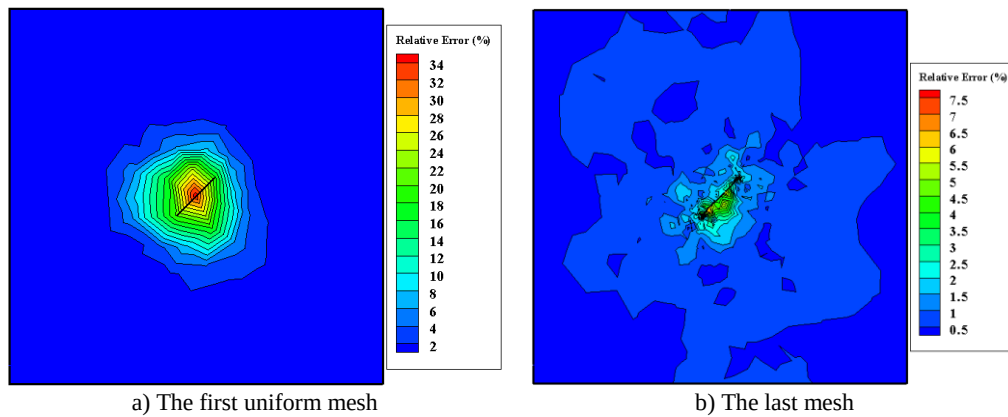


Fig. 21 The relative error distribution for a plate with a central inclined crack

It should be noted that, the initial mesh of the second test case is very coarse (Fig. 15(a)), Then accuracy of the first stress analysis is very low (Fig. 22), so that location of the most relative error is determined where other than the

crack tip. Of course, this issue is resolved after the first refinement.

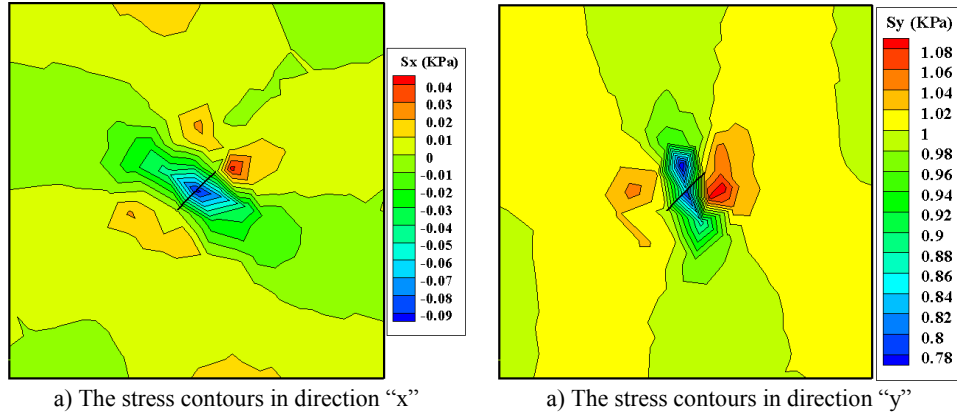


Fig. 22 The stress contours computed on the initial mesh

7.2 Comparison of the accuracy and efficiency of three numerical methods

As can be seen in Fig. 23, the XFEM takes less mean percent error than other methods, so that the mean percent error of the XFEM, adaptive GFEM and adaptive GFVM are 0.75%, 2.5% and 2.1% in test case 1 (Section 7.1.1) and 0.97%, 1.43% and 1.86% in test case 2 (Section 7.1.2).

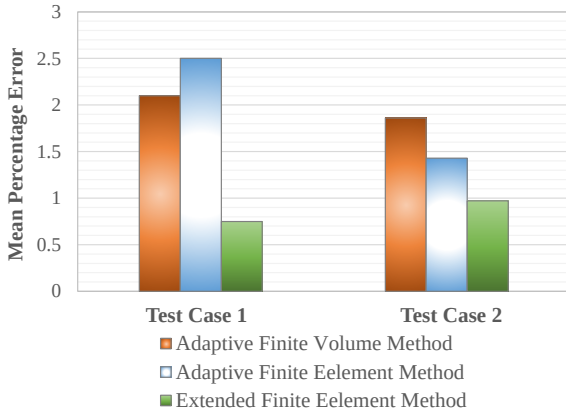


Fig 23. Comparison of mean errors (percent) between different numerical methods

As can be seen in Fig. 24, the adaptive GFVM takes less CPU time than other methods, so that the CPU time of the adaptive GFVM, adaptive GFEM and XFEM are 3.7, 11 and 15.8 seconds in test case 1 and 3, 9 and 22 seconds in test case 2. This difference is attributed to matrix-free algorithm of the GFVM and matrix base procedure of the finite element methods.

The XFEM and adaptive GFEM require matrix operation, therefore CPU time is significantly prolonged by increasing the number of elements. But, the GFVM is a matrix-free method, therefore it does not need extensive computation which leads to significant reduction in the CPU time. Also in the adaptive GFVM method, the computed displacement fields from the last mesh is used as the initial displacement in the next stress analysis on the

refined mesh which cause this iterative method to converge more rapidly.

At first glance, this time difference may not be significant, but the fracture analysis includes iterative calculations which drastically increases the computational time.

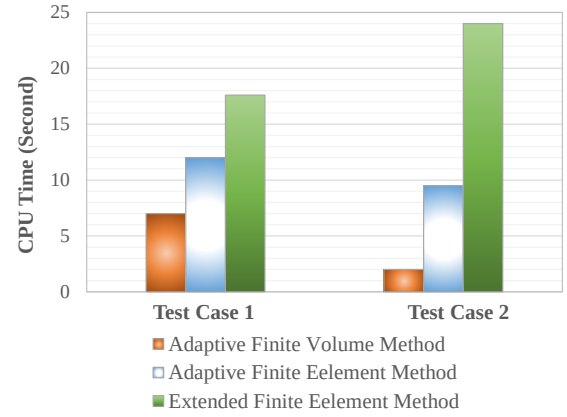


Fig 24. Comparison of CPU time between different numerical methods

8. Conclusion

In this research, the GFVM along with the h-refinement adaptive process and post-processing error estimation analysis is used to 2D fracture analysis. Then two versions of the Finite Element Method known in solid mechanics (Adaptive GFEM and XFEM) are compared with the adaptive GFVM to investigate the accuracy and efficiency of the developed adaptive GFVM. The performance of these numerical methods in fracture mechanics has been compared in terms of accuracy and efficiency (CPU time consumption).

First, the governing equations are discretized according to the numerical methods, then the resulting equations are implemented in FORTRAN. For both the adaptive Finite Element and Finite Volume Methods, similar error estimation and adaptive refinement technique are employed until a nearly uniform computational error is obtained over the entire domain. The crack is modeled

through the enrichment of the displacement approximation around the crack surface in the XFEM. The interaction integral is used to calculate the stress intensity factors.

After ensuring the accuracy of the developed numerical models, the fracture analysis is performed using the above-mentioned three methods for a plate under pure mode I and mixed mode I/II. Generally, the numerical results of various test cases approved the stability and accuracy of the three numerical methods for the fracture analysis of two-dimensional isotropic problems.

The comparison of the results of the three methods show that, the XFEM predicts the stress intensity factors with higher accuracy (The average percent error of the XFEM, adaptive GFEM and adaptive GFVM is 0.88%, 2% and 1.75%, respectively), but this method requires more computational time due to more DOFs and larger matrices in their discretized formulations. On the other hand, the minimum computational time is achieved using the adaptive GFVM (in the order of 5.5 and 3 times less than the XFEM and adaptive GFEM).

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