

## Stability Analysis of Non-Local Euler-Bernoulli Beam with Exponentially Varying Cross-Section Resting on Winkler-Pasternak Foundation

M. Soltani\*, and M. Mohammadi\*\*

### ARTICLE INFO

Article history:

Received:

June 2017.

Revised:

September 2017.

Accepted:

December 2017.

Keywords:

Critical buckling load

Nonlocal elasticity theory

Tapered beam

Elastic foundation

Differential quadrature

method

### Abstract:

In this paper, linear stability analysis of non-prismatic beam resting on uniform Winkler-Pasternak elastic foundation is carried out based on Eringen's non-local elasticity theory. In the context of small displacement, the governing differential equation and the related boundary conditions are obtained via the energy principle. It is also assumed that the width of rectangle cross-section varies exponentially through the beam's length while its thickness remains constant. The differential quadrature method as a highly accurate mathematical methodology is employed for solving the equilibrium equation and obtaining the critical buckling load of simply supported beam. Several numerical results are finally provided to demonstrate the effects of different parameters such as elastic foundation modulus, nonlocal Eringen's parameter and tapering ratio on the critical loads of an exponential tapered non-local beam lying on Winkler-Pasternak foundation. The numerical outcomes indicate that the critical loads of pinned-pinned beam decrease by increasing nonlocal parameter. Furthermore, results show that the elastic foundation enhances the stability characteristics of non-local Euler-Bernoulli beam with constant or variable cross-section. It is finally concluded that the effect of non-uniformity in the cross-section plays significant roles on linear stability behavior of non-local beam.

### 1. Introduction

Due to great progress of nanotechnology, micro/nano-sized systems are increasingly being used in different modern engineering applications and biomedical devices. It is thus essential to exactly predict stability capacity and vibration frequencies of these kinds of members. Due to the relevance of micro/nano-sized continuous elements to different industries, there are a large number of experimental researches (Fleck *et al.*, 1994 [9]; Stolken and Eavans, 1998 [24]; Lam *et al.*, 2003 [13]) which have been dedicated to survey the static and dynamic behavior of these structures. There are significant differences between the acquired results from classical continuum theories and experimental tests. This can be explained by the fact that the vibration and buckling characteristics of micro- and nano-scaled elements are strongly size-dependent and this phenomenon is not considered in classical theories in elasticity. It is worth mentioning that the classical local theories assume that the stress at a point is a function of strain at that point. To date, various size dependent continuum theories such as classical couple stress theory, modified couple stress theory, strain gradient elasticity theory, surface energy theory and nonlocal

theory have been expanded. These theories contain information regarding the inter-atomic forces and the material length scale parameter that is introduced into the constitutive equations as a material parameter. In recent years, the nonlocal elasticity theory developed by Eringen, 1983 [8] has been commonly adopted due to its simplicity and competency for modelling of micro/nano scaled mechanical components (Peddieson *et al.*, 2003 [19]; Sudak, 2003 [25]; Reddy, 2007 [21]; Wang and Liew, 2007 [29]; Phadikar and Pradhan, 2010 [20]; Ghannadpour *et al.*, 2013 [10]; Ebrahimi and Salari, 2015 [5]; Ebrahimi and Mokhtari, 2015 [6]; Pandeya and Singhb, 2015 [18]; Torabi *et al.*, 2015 [26]; Rahmanian *et al.*, 2016 [22]; Mercan and Civalek, 2016 [16]; Hosseini Hashemi and Bakhshi Khaniki, 2016 [11]). Based on this theory, the stress at a reference point is a function of the strains at all points in the body. During the recent years, one of the most significant nano/micro sized structures is the nano/micro beams which have received increasing attention in designing nanoactuators, nanowires and micro/nano sensors. Besides, elastic flexural members whose cross-sectional profile change partially or gradually along their length, known as non-prismatic elements, are widely spread in many engineering applications. This is because of their ability in improving both strength and stability of

\* corresponding Author: Assistant Professor, Department of civil engineering, University of Kashan, Iran (msoltani@kashanu.ac.ir)

\*\* MSc Student in Structural Engineering, University of Kashan, Kashan, Iran

structures. Due to improvements in nanotechnology industries and combination of the advantages of non-prismatic beam and nano-material properties, designers are capable of producing nano/micro devices more efficiently and with more favorable strength. The investigations of elastic buckling load and natural frequency of members lying on elastic foundations is further one of the complicated and significant problems in different fields of structural, mechanical and foundation engineering. In this regard, different types of elastic foundation models like Winkler, Pasternak and Vlasov were presented. The Winkler-type elastic foundation is the most popular mechanical model used to solve the problems mentioned above. In this model, the elastic foundation is considered as the limiting case of an infinitely dense distribution of translational springs with linear behavior, which are independent of each other. However, modelling of the elastic foundation by Winkler's theory was found to be inadequate regarding several problems, since this model overlooks cohesion of the medium. In order to improve this weakness, various two-parameter elastic foundation models such as Winkler-Pasternak foundation were developed. In this model, an additional layer is considered in the widely used Winkler model in order to accomplish the effect of shear interactions between the springs.

Among the first investigations on this topic, the most important one is the study of Reddy, 2007 [21], in which the analytical solutions of bending, buckling and vibration for nonlocal differential elasticity approach of various beam theories are developed. Wang *et al.*, 2007 [28] investigated the bending vibration problem of micro- and nanobeams based on the Eringen's nonlocal elasticity theory and Timoshenko beam assumptions. By using the nonlocal continuum rod model, axial vibration of nanorod with various end conditions was investigated by Aydogdu, 2009 [2]. The free vibration and bending of cantilever microtubules with nonlocal continuum model and fixed-free boundary condition were surveyed by Civalek *et al.*, 2010 [4]. Phadikar and Pradhan, 2010 [20] introduced a finite element solution for nanobeams and nanoplates using the nonlocal differential constitutive relations of Eringen. Based on nonlocal Timoshenko beam theory, stability analysis of nanotubes embedded in an elastic matrix was also performed by Wang *et al.*, 2012 [30]. Akgoz and Civalek, 2013 [1] calculated linear buckling response of linearly tapered micro-columns having different taper ratios via Rayleigh-Ritz method. The surface effects on the nonlinear free vibration of elastically restrained non-local beams with variable cross-section were examined by Malekzadeh and Shojaee, 2013 [15]. Ritz method was utilized by Ghannadpour *et al.*, 2013 [10] to investigate the bending, buckling and vibration of nonlocal Euler beam with arbitrary boundary conditions. Tsias, 2014 [27] presented a new influential approach to exactly determine stiffness and mass matrices of non-uniform Euler-Bernoulli beam from inhomogeneous linearly elastic material resting on an elastic foundation. An investigation on transverse vibration characteristics of rotating functionally graded Timoshenko size-dependent nanobeams made of porous as well as functionally graded material via the semi-analytical differential transformation

method was accomplished by Ebrahimi and Salari, 2015 [5] and Ebrahimi and Mokhtari, 2015 [6]. A finite element solution was proposed by Pandeya and Singhb, 2015 [18] to survey free vibration behavior of fixed-free nanobeam with varying cross-section. Free vibration of Timoshenko nanobeams with varying cross-section was analyzed by Torabi *et al.*, 2015 [26] by adopting generalized differential quadrature method. Rahmanian *et al.*, 2016 [22] analyzed free vibration of carbon nanotube on elastic foundation. Mercan and Civalak, 2016 [16] employed discrete singular convolution technique to obtain exact solution of critical loads for boron nitride nanotube (BNNT) on elastic matrix. Additionally, small scale effects on transverse free vibration of exponentially tapered nanobeams was researched by Hosseini Hashemi and Khaniki, 2016 [11]. Recently, based on various nonlocal higher order shear deformation beam theories, accurate analytical solutions were proposed by Refaieejad *et al.*, 2017 [23] for bending, buckling, and free vibration analyses of functionally graded nanobeam resting on Winkler-Pasternak elastic foundation.

In this paper, we aim to use the differential quadrature method (DQM) for linear stability analysis of exponentially tapered non-local beams lying on an elastic foundation. The gist of the paper is presented below:

1- Based on Euler-Bernoulli beam assumption and using non-local elasticity theory, the linear equilibrium equations are derived from the energy principle for exponential tapered beams related to constant compressive axial load and resting on uniform Winkler-Pasternak type foundation. Due to the complicated mathematical structure of the resulting fourth-order differential equation, closed-form solutions are not accessible. In order to overcome this difficulty, the differential quadrature method is adopted.

2- Regarding the differential quadrature (DQ) rules, the resulting differential equation and the related boundary conditions are discretized and formulated at pre-specified discrete points in the longitudinal direction. Afterwards, the governing equation and end conditions are reduced into a set of linear simultaneous algebraic equations. The smallest real root to the obtained algebraic equation is considered as critical buckling load.

According to the steps mentioned above, several illustrative examples are represented to measure the effects of non-uniformity ratio, Winkler-Pasternak foundation modulus and non-local parameter on critical buckling loads of simply supported non-local Euler-Bernoulli beam. Comments and conclusions are presented towards the end of the manuscript.

## 2. Derivation of the governing equations

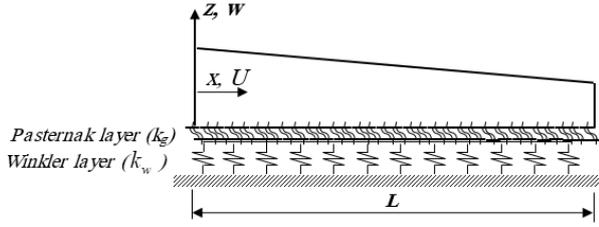
Consider a straight tapered beam element of length span  $L$  resting on uniform Winkler-Pasternak elastic foundation (Fig. 1) and subjected to a compressive axial load  $P$ . We consider the right hand Cartesian co-ordinate system, with  $x$  the initial longitudinal axis measured from the left end of the beam, the  $y$ -axis in the lateral direction and the  $z$ -axis along the thickness of the beam. The origin of these axes ( $O$ ) is located at the centroid of the cross-section. It should

be noted that the Euler-Bernoulli beam assumptions are adopted. According to this theory, the effect of shear deformation is neglected and only the influence of flexural deformation is taken into consideration in the calculation process. Based on Euler beam theory, the longitudinal and transverse displacement components can be respectively expressed as:

$$U(x, y, z) = -zw'(x) \quad (1a)$$

$$W(x, y, z) = w(x) \quad (1b)$$

In the previous equations,  $U$  denotes the axial displacement and  $W$  signifies the vertical displacement (in  $z$  direction).



**Fig. 1:** Tapered beam resting on two-parameter elastic foundation

The equilibrium equations for Euler beam with variable cross-section are derived if the first variation of the total potential energy vanishes:

$$\delta\Pi = \delta(U_l + U_0 + U_f - W_e) = 0 \quad (2)$$

$\delta$  illustrates a virtual variation in the last formulation.  $U_l$  represents the elastic strain energy,  $U_0$  expresses the strain energy due to effects of the initial stresses.  $U_f$  is the energy corresponding to uniform elastic foundation and  $W_e$  denotes the work of applied loads. For the particular case of linear stability context, where the beam is not under any external force, one considers that the external load work equals to zero.  $\delta\Pi$  could be computed using the following equation (Mercan and Civalek, 2016 [16]):

$$\delta\Pi = \int_0^L \int_A (\sigma_{xx} \delta\varepsilon_{xx}^l) dA dx + \int_0^L \int_A (\sigma_{xx}^0 \delta\varepsilon_{xx}^*) dA dx + \int_0^L (k_w w \delta w + k_g w' \delta w') dx \quad (3)$$

In which,  $L$  and  $A$  express the element length and the cross-sectional area, respectively.  $(\delta\varepsilon_{xx}^l, \delta\gamma_{xz}^l)$  and  $\delta\varepsilon_{xx}^*$  are the variation of the linear and the non-linear parts of strain tensor, respectively.  $k_w$  and  $k_g$  denote Winkler elastic foundation constant and the second foundation parameter modulus in vertical direction, respectively.  $\sigma_{xx}$  is axial stress and  $\sigma_{xx}^0$  signifies initial normal stress in the cross-section, associated with constant axial force ( $P$ ):

$$\sigma_{xx}^0 = -\frac{P}{A} \quad (4)$$

Based on the assumption of the Green's strain-tensor, the strain-displacement relations including the linear and the non-linear parts are:

$$\varepsilon_{xx}^l = -zw''(x) \quad (5a)$$

$$\varepsilon_{xx}^* = \frac{1}{2}(w'(x))^2 \quad (5b)$$

According to the last formulations, the variation of the strain tensor components is given by:

$$\delta\varepsilon_{xx}^l = -z \delta w''(x) \quad (6a)$$

$$\delta\varepsilon_{xx}^* = w'(x) \delta w'(x) \quad (6b)$$

Substituting equations (4) to (6) into relation (3), the expression of the virtual potential energy can be carried out as:

$$\begin{aligned} \delta\Pi &= \int_0^L \int_A (\sigma_{xx} (-z \delta w''(x))) dA dx \\ &+ \int_0^L \int_A \left( -\frac{P}{A} (w' \delta w') \right) dA dx \\ &+ \int_0^L (k_w w \delta w + k_g w' \delta w') dx = 0 \end{aligned} \quad (7)$$

The variation of strain energy can be formulated in terms of section forces acting on cross-sectional contour of the elastic member in the buckled configuration. The section stress resultants are presented by the following expressions:

$$N = \int_A \sigma_{xx} dA \quad (8a)$$

$$M = \int_A \sigma_{xx} z dA \quad (8b)$$

$N$  and  $M$  are the axial force applied at end member and the bending moment, respectively. Using relations (7)–(8), the final form of the total potential energy variation ( $\delta\Pi$ ) is then acquired as:

$$\begin{aligned} \delta\Pi &= -\int_0^L (M \delta w''(x)) dx - \int_0^L (P w' \delta w') dx \\ &+ \int_0^L (k_w w \delta w + k_g w' \delta w') dx = 0 \end{aligned} \quad (9)$$

According to the equation presented above, the first variation of strain energy contains the virtual displacement ( $\delta w$ ) and its derivatives. After appropriate integrations by parts, one gets an expression in terms of virtual displacement. After some calculations and essential simplifications, the following equilibrium equation in the stationary state is obtained:

$$\frac{d^2 M}{dx^2} = (P - k_g) \frac{d^2 w}{dx^2} + k_w w \quad (10)$$

The boundary conditions of the present beam theory can also be expressed as:

$$M = 0 \quad \text{Or} \quad w' = 0 \quad (11a)$$

$$M' - (P - k_g) w' = 0 \quad \text{Or} \quad w = 0 \quad (11b)$$

According to the Eringen non-local elasticity model (Eringen, 1983 [8]), the stress at a point depends not only on the strain state at that point but also on strain states at all other points throughout the body. For one dimensional elastic material, the nonlocal constitutive relation can be written as:

$$\sigma_{xx} - \mu \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E \varepsilon_{xx}^l \quad (12)$$

Where  $E$  is the Young's moduli.  $\mu=(e_0a)^2$  denoting the non-local parameter;  $e_0$  is a material constant which is determined experimentally or approximated by matching the dispersion curves of plane waves with those of atomic lattice dynamics; and  $a$  is an internal characteristic length of the material (e.g., lattice parameter, C–C bond length and granular distance). In general, the nonlocal parameter is  $0<\mu<4.0 \text{ nm}^2$  for a single wall carbon nanotubes (SWCNTs) [29]. It is important to note that the constitutive relation for the shear stress and strain remains the same as in the local beam theory for homogeneous isotropic one-dimensional case (Eringen and Suhubi, 1964 [7]; Eringen, 1983 [8]).

Multiplying Eq. (12) by  $zdA$  and integration over the cross-sectional area in the context of principal axes, the following expression is obtained:

$$M(x) - \mu \frac{d^2M}{dx^2} + EI \frac{d^2w}{dx^2} = 0 \quad (13)$$

In Eq. (13),  $I$  signifies the minor moment of inertia about  $y$  axis. The following bending moment expression for the nonlocal beam theory is acquired by substituting the governing equation (10) into (13):

$$M(x) = \mu((P - k_g) \frac{d^2w}{dx^2} + k_w w) - EI \frac{d^2w}{dx^2} \quad (14)$$

By substituting Eq. (14) into Eq. (10), the nonlocal equilibrium equation can be expressed in terms of vertical displacement ( $w$ ) as:

$$\begin{aligned} & \frac{d^2}{dx^2} (EI \frac{d^2w}{dx^2}) - \mu(P - k_g) \frac{d^4w}{dx^4} \\ & + (P - k_g) \frac{d^2w}{dx^2} - \mu k_w \frac{d^2w}{dx^2} + k_w w = 0 \end{aligned} \quad (15)$$

Extending the above equation results in:

$$\begin{aligned} & EI \frac{d^4w}{dx^4} + 2E \frac{dI}{dx} \frac{d^3w}{dx^3} + E \frac{d^2I}{dx^2} \frac{d^2w}{dx^2} \\ & - \mu(P - k_g) \frac{d^4w}{dx^4} + (P - k_g) \frac{d^2w}{dx^2} \\ & - \mu k_w \frac{d^2w}{dx^2} + k_w w = 0 \end{aligned} \quad (16)$$

In this study, it is assumed that the width of beam's section ( $b_0$ ) at the left support is made to diminish to ( $b_1 = b_0 e^{-\alpha}$ ) at the other end with an exponential variation, while its thickness ( $h_0$ ) remains constant (Fig. 2). The variation of minor axis moment of inertia and cross-sectional area along the length of the beam are thus described as follows:

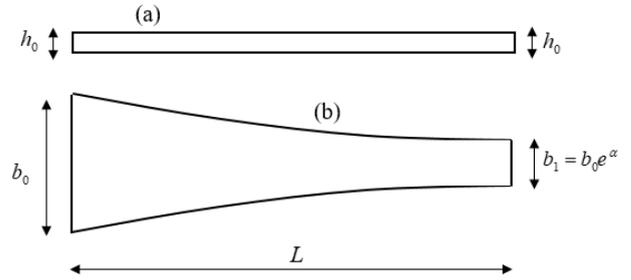
$$I(x) = I_0 e^{\frac{\alpha x}{L}} \quad (17a)$$

$$A(x) = A_0 e^{\frac{\alpha x}{L}} \quad (17b)$$

The exponential non-uniformity parameter ( $\alpha$ ) can change from zero (prismatic beam) to a range of  $[-2$  to  $-0.1]$  for non-uniform beams.  $A_0$  and  $I_0$  are respectively cross-sectional

area and moment of inertia at the left support ( $x=0$ ). They are defined as:

$$I_0 = \frac{b_0 h_0^3}{12} \text{ and } A_0 = b_0 h_0 \quad (18)$$



**Fig. 2:** Schematic representation of an exponential tapered beam, (a) Constant thickness through beam's length, (b) Exponentially varying width along member.

Substituting Eq. (17) into (16), the equilibrium equation of non-local beam with varying cross-section could be rewritten as:

$$\begin{aligned} & EI_0 e^{\frac{\alpha x}{L}} \frac{d^4w}{dx^4} + 2EI_0 \left(\frac{\alpha}{L}\right) e^{\frac{\alpha x}{L}} \frac{d^3w}{dx^3} \\ & + EI_0 \left(\frac{\alpha}{L}\right)^2 e^{\frac{\alpha x}{L}} \frac{d^2w}{dx^2} - \mu(N - k_g) \frac{d^4w}{dx^4} \\ & + (N - k_g) \frac{d^2w}{dx^2} - \mu k_w \frac{d^2w}{dx^2} + k_w w = 0 \end{aligned} \quad (19)$$

Exponential variation of geometrical properties is one of the most special cases of non-prismatic member in which, few numerical techniques are able to solve the governing equilibrium equation. In this regard, closed-form solutions of the differential equation (19) are not accessible. In the following, the DQM is adopted to solve the equilibrium equation of exponentially tapered beams with non-local theory. This methodology could be effectively tuned to determine the critical buckling loads of beams whose geometrical properties vary exponentially along the beam.

### 3. DQM formulation of the problem

In the presence of arbitrary variation in geometrical properties, the governing equilibrium equation (Eq. (19)) becomes a differential equation with variable coefficients in which the classical methods used in stability analysis of prismatic members are not efficient and no longer valid. For such complicated problems, the differential quadrature method which was first proposed by Bellman and Casti, 1971 [3], is employed to solve the resulting fourth-order differential equation (Eq. (19)). The basic concept of the differential quadrature method is to discretize the derivatives of a function with respect to a variable in the differential equation at a sample point as a weighted linear summation of function values at its adjacent points. Governing equations and external boundary conditions are thus transformed into a set of linear algebraic equations, which can be solved with the aid of a computational algorithm to derive an approximate solution of the continuous differential equations. For this, it is first

required to divide the computational region into  $N$  discrete nodes or list of sampling points spanning the domain of solution. Hence, the precision of this numerical approach depends on the number and type of adopted sampling points. One of the best selection of the sampling points in the stability and vibration analyses is the Chebyshev-Gauss-Lobatto points:

$$x_i = \frac{L}{2} \left[ 1 - \cos \left( \frac{i-1}{N-1} \pi \right) \right], \quad (20)$$

if  $0 \leq x \leq L \quad i = 1, 2, \dots, N$

Where  $N$  is the number of grid points in the longitudinal direction. With this coordinate of the mesh points, the weighting coefficients can be easily determined. In order to facilitate the solution of the stability equation by means of the differential quadrature approach, a non-dimensional variable ( $\xi = x / L$ ) is introduced. The expressions of extended form of the equilibrium equation and Eq. (19) can be thus transformed to a non-dimensional form as follows:

$$\begin{aligned} & (EI_0 e^{\alpha \xi}) \frac{d^4 w}{d \xi^4} + 2(E \alpha I_0 e^{\alpha \xi}) \frac{d^3 w}{d \xi^3} + (E \alpha^2 I_0 e^{\alpha \xi}) \frac{d^2 w}{d \xi^2} \\ & + k_g (\mu \frac{d^4 w}{d \xi^4} - L^2 \frac{d^2 w}{d \xi^2}) + k_w (L^4 w(\xi) - \mu L^2 \frac{d^2 w}{d \xi^2}) \\ & + P (L^2 \frac{d^2 w}{d \xi^2} - \mu \frac{d^4 w}{d \xi^4}) = 0 \end{aligned} \quad (21a)$$

$$\xi_i = \frac{1}{2} \left[ 1 - \cos \left( \frac{i-1}{N-1} \pi \right) \right], \quad (21b)$$

if  $0 \leq \xi \leq 1 \quad i = 1, 2, \dots, N$

At a given grid point  $\xi_i$ , the first order derivative of vertical deflection ( $w$ ) can be approximated as:

$$w'(\xi) \Big|_{\xi=\xi_i} = \sum_{j=1}^N A_{ij}^{(1)} w(\xi_j) \quad (22)$$

Where  $w(\xi_j)$  is function values at grid points  $\xi_j$  ( $i = 1, 2, \dots, N$ ).  $A_{ij}^{(1)}$  denotes the weighting coefficient for the first order derivative of deflection. This coefficient is computed by the following algebraic formulations which are based on Lagrangian interpolation polynomials:

$$A_{ij}^{(1)} = \begin{cases} \frac{M(\xi_i)}{(\xi_i - \xi_j) M(\xi_j)} & \text{for } i \neq j \\ - \sum_{k=1, k \neq i}^N A_{ik}^{(1)} & \text{for } i=j \end{cases} \quad i, j = 1, 2, \dots, N \quad (23a)$$

Where,

$$M(\xi_i) = \prod_{j=1, j \neq i}^N (\xi_i - \xi_j) \quad \text{for } i = 1, 2, \dots, N \quad (23b)$$

Higher order derivatives of displacement at each grid point can be acquired from the first order weighting coefficient as follows:

$$w''(\xi) \Big|_{\xi=\xi_i} = \sum_{j=1}^N A_{ij}^{(2)} w(\xi_j), \quad \sum_{j=1}^N A_{ij}^{(2)} = \sum_{k=1}^N A_{ik}^{(1)} A_{kj}^{(1)} \quad (24a)$$

$$w'''(\xi) \Big|_{\xi=\xi_i} = \sum_{j=1}^N A_{ij}^{(3)} w(\xi_j), \quad \sum_{j=1}^N A_{ij}^{(3)} = \sum_{k=1}^N A_{ik}^{(1)} A_{kj}^{(2)} \quad (24b)$$

$$w^{IV}(\xi) \Big|_{\xi=\xi_i} = \sum_{j=1}^N A_{ij}^{(4)} w(\xi_j), \quad \sum_{j=1}^N A_{ij}^{(4)} = \sum_{k=1}^N A_{ik}^{(1)} A_{kj}^{(3)} \quad (24c)$$

Applying the differential quadrature discretization to the non-dimensional governing equation (Eq. (21a)) leads to the following expression:

$$\begin{aligned} & (EI_0 e^{\alpha \xi_j}) \left( \sum_{j=1}^N A_{ij}^{(4)} w_j \right) + 2(E \alpha I_0 e^{\alpha \xi_j}) \left( \sum_{j=1}^N A_{ij}^{(3)} w_j \right) \\ & + (E \alpha^2 I_0 e^{\alpha \xi_j}) \left( \sum_{j=1}^N A_{ij}^{(2)} w_j \right) \\ & + k_g (\mu \sum_{j=1}^N A_{ij}^{(4)} w_j - L^2 \sum_{j=1}^N A_{ij}^{(2)} w_j) \\ & + k_w (L^4 w_j - \mu L^2 \sum_{j=1}^N A_{ij}^{(2)} w_j) \\ & + P (L^2 \sum_{j=1}^N A_{ij}^{(2)} w_j - \mu \sum_{j=1}^N A_{ij}^{(4)} w_j) = 0 \end{aligned} \quad (25)$$

In the current study, a simply supported beam is surveyed. For this, the vertical displacement and bending moment at both ends are prevented. The internal bending moment equation for Euler-Bernoulli non-local beam theory may be rewritten in the following statement in the form of DQM as:

$$\begin{aligned} M &= -(EI_0 e^{\alpha \xi_j}) \left( \sum_{j=1}^N A_{ij}^{(2)} w_j \right) + P \mu \left( \sum_{j=1}^N A_{ij}^{(2)} w_j \right) \\ & + (\mu k_g \sum_{j=1}^N A_{ij}^{(2)} w_j + L^2 k_w w_j) \end{aligned} \quad (26)$$

It is also possible to express the quadrature analog of the mentioned above formulations in the following matrix forms:

$$\begin{aligned} & [a][A]^{(4)} \{w\} + [b][A]^{(3)} \{w\} + [c][A]^{(2)} \{w\} \\ & + k_g (\mu [A]^{(4)} - L^2 [A]^{(2)}) \{w\} \\ & + k_w (L^4 - \mu L^2 [A]^{(2)}) \{w\} \\ & + P (L^2 [A]^{(2)} - \mu [A]^{(4)}) \{w\} = 0 \end{aligned} \quad (27a)$$

$$\begin{aligned} M &= -[a][A]^{(2)} \{w\} + P \mu [A]^{(2)} \{w\} \\ & + (\mu k_g [A]^{(2)} + L^2 k_w) \{w\} \end{aligned} \quad (27b)$$

In which:

$$a_{jk} = EI_0 e^{\alpha \xi_j} \delta_{jk} \quad (28a)$$

$$b_{jk} = 2\alpha EI_0 e^{\alpha \xi_j} \delta_{jk} \quad (28b)$$

$$c_{jk} = \alpha^2 EI_0 e^{\alpha \xi_j} \delta_{jk} \quad (28c)$$

$$\{w\} = \{w(\xi_1) \quad w(\xi_2) \quad w(\xi_3) \quad \dots \quad w(\xi_n)\} \quad (28d)$$

In order to acquire the critical loads more easily, Eq. (27a) can be transformed into the following eigenvalue problem:

$$([K] + [K_S] + P[K_G])\{w\} = 0 \quad (29)$$

In which,  $K$ ,  $K_S$  and  $K_G$  are  $N \times N$  matrices. As mentioned previously,  $N$  denotes the number of grid points along the computation domain ( $0 \leq \xi \leq 1$ ). These terms are thus determined in the following forms:

$$[K] = [a][A]^{(4)} + [b][A]^{(3)} + [c][A]^{(2)} \quad (30a)$$

$$[K_G] = L^2 [A]^{(2)} - \mu [A]^{(4)} \quad (30b)$$

$$[K_S] = k_g (\mu [A]^{(4)} - L^2 [A]^{(2)}) + k_w (L^4 - \mu L^2 [A]^{(2)}) \quad (30c)$$

Additionally, the nonlocal boundary condition is expressed as:

$$M = [R]\{w\} + P[R_G]\{w\} + [R_S]\{w\} \quad (31a)$$

$$[R] = -[a][A]^{(2)} \quad (31b)$$

$$[R_G] = \mu [A]^{(2)} \quad (31c)$$

$$[R_S] = \mu k_g [A]^{(2)} + L^2 k_w \quad (31d)$$

For numerically implementing the natural boundary conditions, Eq. (27b) is transformed to the following expressions:

$$\text{at } \xi = 0 \Rightarrow M_0 = [R_0]\{w\} + P[R_{G0}]\{w\} + [R_{S0}]\{w\} \quad (32a)$$

$$\text{at } \xi = 1 \Rightarrow M_1 = [R_1]\{w\} + P[R_{G1}]\{w\} + [R_{S1}]\{w\} \quad (32b)$$

Therefore, Eq. (27a) is converted to:

$$([K_t] + [K_{St}] + P[K_{Gt}])\{w\} = 0 \quad (33a)$$

$$[K_t] = \begin{bmatrix} K \\ R_0 \\ R_1 \end{bmatrix}; \quad [K_{St}] = \begin{bmatrix} K_S \\ R_{S0} \\ R_{S1} \end{bmatrix}; \quad [K_{Gt}] = \begin{bmatrix} K_G \\ R_{G0} \\ R_{G1} \end{bmatrix} \quad (33b)$$

In the case of pinned-pinned members, the vertical displacement at both ends is equal to zero. This boundary condition can be easily contemplated by eliminating the first and Nth rows and columns of the obtained matrices ( $[K_t]$ ,  $[K_{St}]$  and  $[K_{Gt}]$ ). Finally, we have:

$$([\bar{K}_t] + [\bar{K}_{St}] + P[\bar{K}_{Gt}])\{\bar{w}\} = 0 \quad (34a)$$

$$\{\bar{w}\} = \{w_2 \quad w_3 \quad w_4 \quad \dots \quad w_{n-1}\} \quad (34b)$$

Afterwards, the critical buckling loads for exponentially tapered non-local beam lying on uniform Winkler-Pasternak foundation can be computed from the eigenvalues of Eq. (34a).

#### 4. Numerical results

In the previous sections, the equilibrium equation of non-local beam with exponentially varying width and lying on two-parameter elastic foundation was formulated and numerically solved for linear stability analysis. In order to demonstrate the effects of three different factors namely: nonlocal Eringen's parameter, tapering ratio and elastic foundation on the axial critical load, several illustrative examples are provided in the current section and solved via the proposed mathematical methodology. The following non-dimensional parameters relating to Winkler and Pasternak constants as well as critical load are used to ease and simplify presentation of the acquired numerical values:

$$\bar{k}_w = k_w \frac{L^4}{E_0 J_0} \quad (35a)$$

$$\bar{k}_G = k_G \frac{L^2}{E_0 J_0} \quad (35b)$$

$$P_{nor} = \frac{P_{cr} L^2}{E_0 J_0} \quad (35c)$$

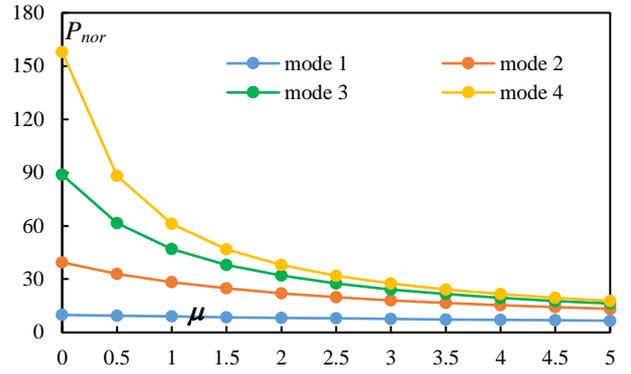
In the present study, the non-dimensional Winkler and Pasternak modulus parameters for uniform two-parameter foundation are taken into account in the range of 0–100 and 0–10, respectively (Liew *et al.*, 2006 [14]; Murmu and Pradhan, 2009 [17]; Khajansari *et al.*, 2012 [12]).

The aim of the first part of the current section is to define the needed number of points along the longitudinal direction while using DQM to obtain an acceptable accuracy on critical elastic buckling loads. Regarding this, Table 1 gives the first non-dimensional buckling load parameters ( $P_{nor}$ ) of simply supported prismatic beams with non-local theory. The convergence study is carried out for various values of non-local parameter. The effects of the number of sampling points used in DQM on convergence are also displayed in Table 1. The obtained results by the proposed numerical technique have been compared with the closed-form solution introduced by Reddy, 2007 [21]. It is clearly seen from Table 1 that twenty number of grid points ( $N=20$ ) are sufficient to obtain the lowest buckling load parameters for different nonlocal parameters with desired accuracy.

After noticing the results presented in Table 1, it can also be concluded that the elastic buckling loads calculated by employing local theory ( $\mu=0$ ) are overestimated. Regarding this, a decrease in buckling parameter of 33 percent is evident for the increase of nonlocal parameter from 0 to 5. This statement can be explained by the fact that the flexural stiffness of pinned-pinned Euler-Bernoulli beam with non-local theory is inversely proportional to the Eringen's parameter. In general, the inclusion of the nonlocal effect increases the deflection, which in turn leads to a noticeable decrease in the value of the stiffness and rigidity of the member and consequently a weaker member is obtained. Since the linear buckling resistance of beam is directly proportional to the stiffness of the member, a significant decrease in the critical load of the beam is thus observed.

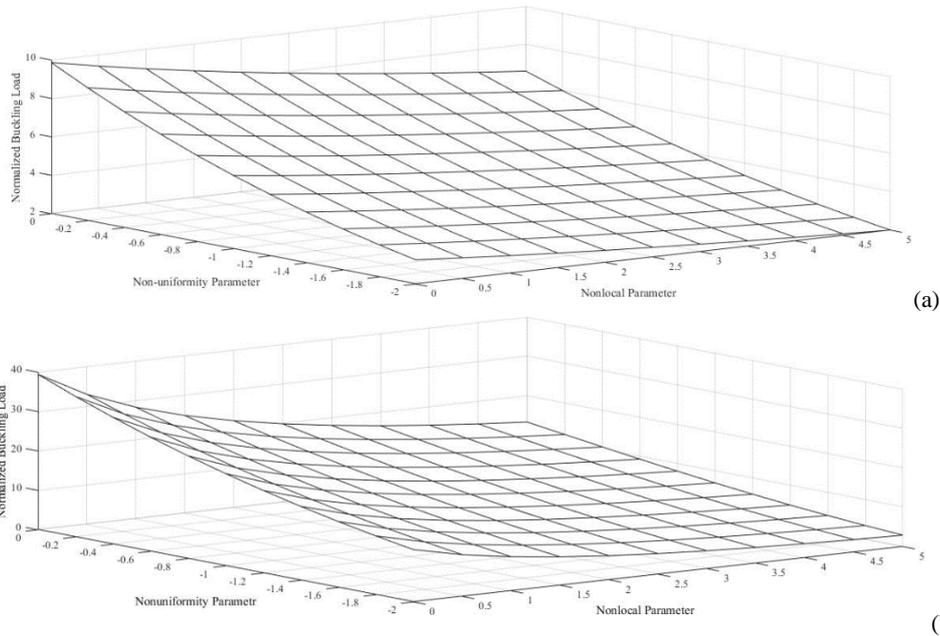
**Table 1:** Convergence of the differential quadrature technique in determination of the lowest non-dimensional critical buckling load parameters ( $P_{nor}$ ) for uniform beam with different non-local parameters

$(\mu)$	DQM					Reddy, 2007 [21]
	Number of points along x-direction					
	5	10	15	20	30	
0.0	9.9678	9.8696	9.8696	9.8696	9.8696	9.8696
0.5	9.4946	9.4055	9.4055	9.4055	9.4054	9.4055
1	9.0643	8.9830	8.9830	8.9830	8.9830	8.9830
1.5	8.6713	8.5969	8.5969	8.5969	8.5969	8.5969
2.0	8.3109	8.2426	8.2426	8.2426	8.2426	8.2426
2.5	7.9794	7.9163	7.9163	7.9163	7.9163	7.9163
3.0	7.6732	7.6149	7.6149	7.6149	7.6149	7.6149
3.5	7.3897	7.3356	7.3356	7.3356	7.3356	7.3356
4.0	7.1264	7.0761	7.0761	7.0761	7.0761	7.0761
4.5	6.8812	6.8343	6.8343	6.8343	6.8343	6.8343
5.0	6.6523	6.6085	6.6085	6.6085	6.6085	6.6085



**Fig. 3:** Effect of non-local parameter ( $\mu$ ) on the first four normalized buckling loads of uniform beam

In the next step, the descendant effect of non-local parameter on first four axial critical loads of non-local beam with uniform cross-section is shown in Fig. 3.



**Fig. 4:** Effects of the non-local parameter ( $\mu$ ) on the normalized buckling load of exponential-tapered beam with different tapering ratios: (a) first mode (b) second mode

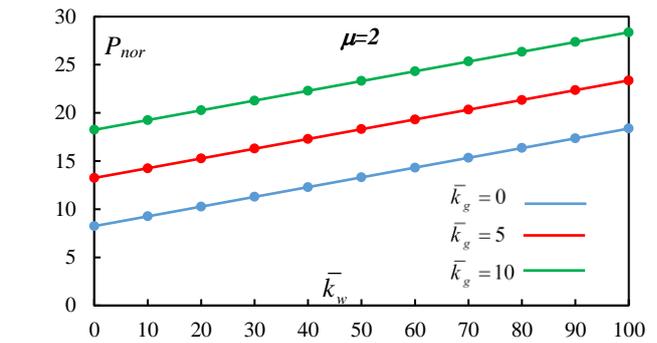
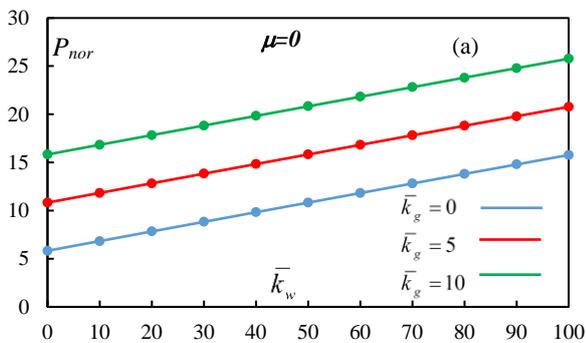
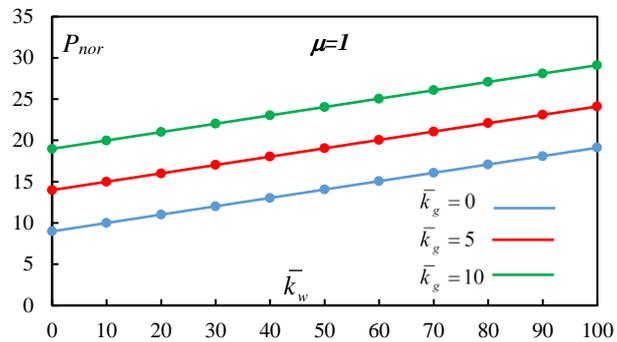
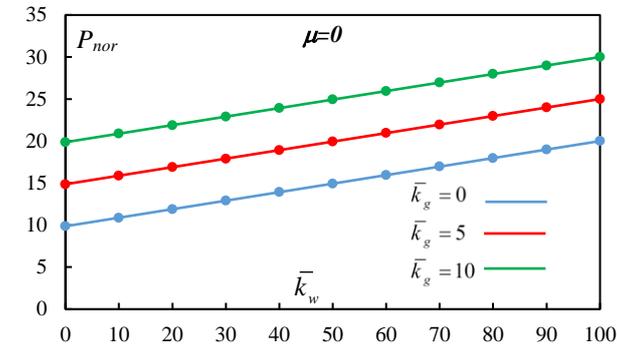
Regarding Fig. 3, similar trends in the outcomes are observed. It is also evident that the non-local parameter has more influence on higher buckling modes compared with the lower ones. It can be stated that, it is necessary to contemplate the non-local theory for exact estimation of critical loads related to higher buckling modes of nano-sized beams.

Following the above-mentioned procedure, the lowest normalized buckling load parameters ( $P_{nor}$ ) for various exponential non-uniformity ratios ( $\alpha$ ) with different nonlocal parameters ( $\mu$ ) are arranged in Table 2.

**Table 2:** Normalized buckling load parameter of exponentially tapered non-local beam for different non-uniformity ratios and non-local parameters

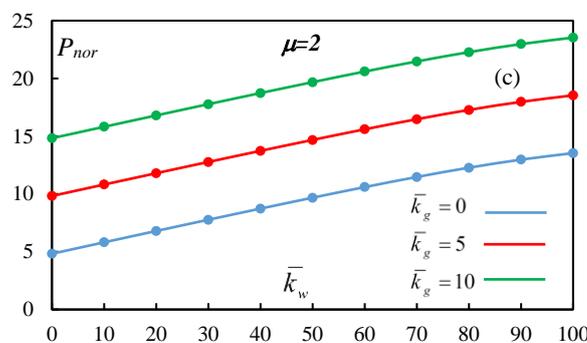
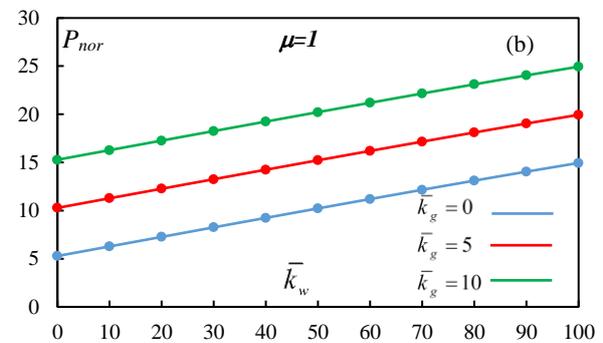
$(\alpha)$	Non-local Parameter ( $\mu$ )					
	0	1.0	2.0	3.0	4.0	5.0
0.0	9.870	8.983	8.243	7.615	7.076	6.608
-0.2	8.921	8.118	7.448	6.879	6.391	5.968
-0.4	8.046	7.318	6.710	6.195	5.752	5.369
-0.6	7.241	6.580	6.028	5.561	5.159	4.811
-0.8	6.502	5.902	5.400	4.975	4.611	4.294
-1.0	5.826	5.280	4.824	4.438	4.106	3.818
-1.2	5.210	4.713	4.298	3.946	3.644	3.382
-1.4	4.649	4.196	3.818	3.497	3.222	2.983
-1.6	4.140	3.727	3.382	3.090	2.839	2.622
-1.8	3.679	3.303	2.988	2.722	2.494	2.295
-2.0	3.263	2.920	2.633	2.390	2.182	2.002

Moreover, Fig. 4 illustrates the variation of the first two normalized buckling loads of exponentially tapered non-local beam with respect to the tapering ratio ( $\alpha$ ) and the non-local parameter ( $\mu$ ) for simply supported beams.

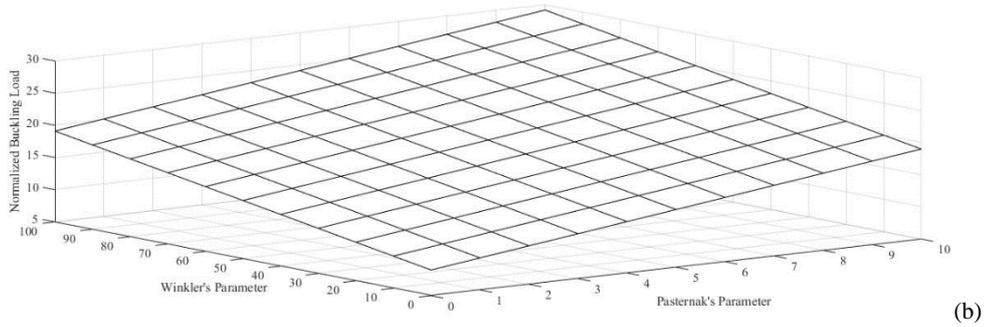
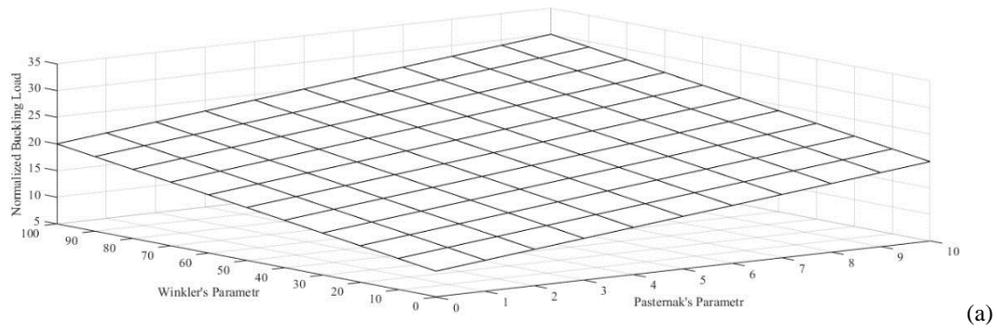


**Fig. 5:** Variation of normalized buckling load of uniform beam versus Winkler type parameter for three different values of Pasternak parameter

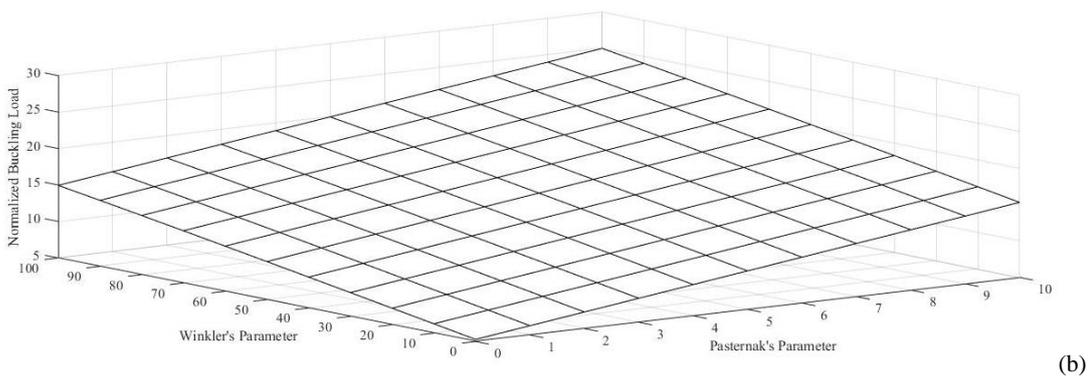
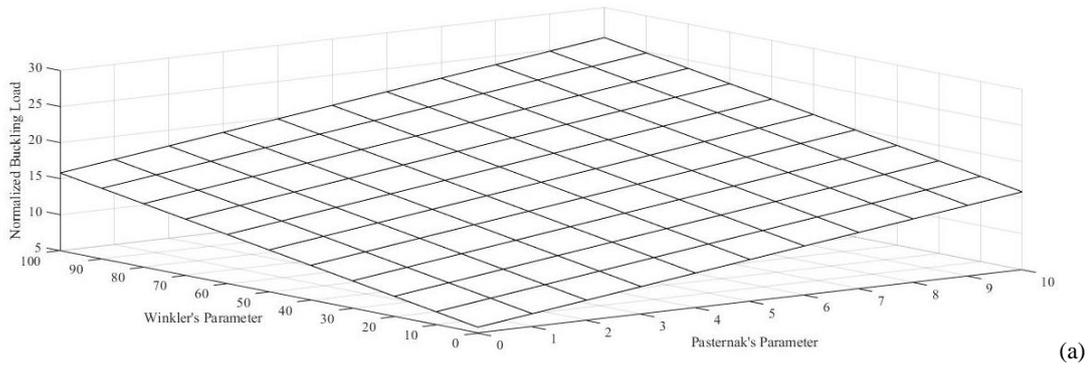
Comparing the results of prismatic beam depicted in Table 1 with those related to non-prismatic ones (Table 2 and Fig. 4), it can be culminated that for any value of non-local parameters, the corresponding buckling load for the beam with uniform cross-section is the highest and that for tapered beam with the non-uniformity ratio (-2) is the lowest. This manifest is reasonable due to the fact that an increase in taper ratios causes reduction in cross-sectional area and moment of inertia and consequently in the stiffness of the elastic member. Moreover, it is easily observed from Fig. 4 that the variation of non-local parameter has an important impact on the buckling capacity of non-uniform beams with pinned-pinned end conditions. It can also be stated that the critical buckling load parameters corresponding to the first mode rapidly diminish as the non-local parameter increases. Similar behavior is observable for the second non-dimensional buckling load parameters (Fig. 4b).



**Fig. 6:** Variation of normalized buckling load of tapered beam ( $\alpha=-1$ ) versus Winkler type modulus for three different values of Pasternak parameter



**Fig. 7:** Variation of normalized buckling load of prismatic beam for different Winkler and Pasternak foundation parameters, (a)  $\mu=0$ , (b)  $\mu=1$



**Fig. 8:** Variation of normalized buckling load of non-uniform beam ( $\alpha=-1$ ) for different Winkler and Pasternak foundation parameters, (a)  $\mu=0$ , (b)  $\mu=1$

In the last section, the influence of uniform Winkler-Pasternak foundation on the linear buckling resistance of uniform and exponentially tapered simply supported beams based on non-local theory are surveyed. In the case

of tapered beam, the non-uniformity parameter is taken to be  $\alpha=-1$ . The variation of the lowest buckling load parameters for uniform and exponentially tapered Euler-

Bernoulli beams versus the elastic foundation constant ( $\bar{k}_w$ ) with three different Pasternak parameter values are respectively presented in Figs. 5 and 6. As presented in Figs. 7 and 8, the simultaneous influence of two foundation constants namely: Winkler and Pasternak on the buckling capacity of the contemplated beams with two different non-local parameters are surveyed.

As can be seen, the variations of Winkler and Pasternak elastic foundation parameters have a significant influence on the linear stability behavior of local and non-local beams under varying circumstances. It is also observed from these illustrations that the Winkler type foundation has a little more impact on the buckling resistance of simply supported beam, in contrast with the Pasternak layer modulus. In addition, it can be culminated from these figures that the critical buckling load parameters corresponding to the first mode are increased as the stiffness of the elastic foundation increases. In other words, the numerical outcomes show that the elastic foundation has a stabilizing effect on the stability characteristics of simply supported non-local beams with constant or variable cross-section.

## 5. Conclusions

In this research, nonlocal elasticity theory and Euler-Bernoulli beam assumptions are adopted to formulate the equilibrium equation of non-uniform beam lying on two-parameter elastic foundation. In the presence of arbitrary variation in cross-section, the governing equilibrium equation becomes a differential equation with variable coefficients in which the classical methods used in stability analysis of prismatic members are not efficient and no longer valid. Differential quadrature method (DQM) is thus utilized for the numerical solution of the resulting fourth-order differential equation and calculation of the axial critical load. The influences of non-local parameter, tapering ratio, Winkler type spring constant and Pasternak type shear constant on the critical buckling load of exponentially tapered non-local beam with pinned-pinned end conditions are studied in detail. From the results of the present study, the following conclusions can be addressed:

- In most cases, it can be concluded that by considering the small number of grid points in DQ approach, the critical loads related to the first few buckling modes of non-local beam with exponentially varying width and resting on elastic foundation can be determined with very good accuracy. Therefore, the current study reveals the power of differential quadrature method in solving differential equations with variable coefficients.
- The increase of the nonlocal parameter causes to decrease the critical buckling loads. The linear stability capacity of non-local beams with constant or variable cross-section is smaller than their local counterparts.
- The numerical outcomes show that the small scale effects are more significant for higher buckling modes, and thus the small scale effect is not ignorable.
- The buckling strength of exponentially tapered beam is smaller than that of beam with uniform cross-section.

- It can be stated that the non-dimensional buckling load parameter for prismatic beam decreases more by increasing non-local parameter with respect to the exponentially tapered beam.

- Finally, it is observed that Winkler and Pasternak constants improve the stability characteristics of non-local beam with constant or variable cross-section.

## References:

- [1] Akgoz B., Civalek O., "Buckling analysis of linearly tapered micro-columns based on strain gradient elasticity", *Structural Engineering and Mechanics*, vol. 48, 2013, p.195–205.
- [2] Aydogdu M., "Axial vibration of the nanorods with the nonlocal continuum rod model", *Physica E.*, vol. 41, 2009, p. 861–864.
- [3] Bellman R.E., Casti J., "Differential quadrature and long-term integration", *Journal of Mathematical Analysis and Applications*, vol. 34, 1971, p. 235-238.
- [4] Civalek O., Demir C., Akgoz B., "Free vibration and bending analyses of cantilever microtubules based on nonlocal continuum model", *Math. Comput. Appl.*, vol. 15, 2010, p. 289–298.
- [5] Ebrahimi F., Salari E., "Size-dependent free flexural vibrational behavior of functionally graded nanobeams using semi-analytical differential transform method", *Composite Part B.*, vol. 79, 2015, p. 156–169.
- [6] Ebrahimi F., Mokhtari M., "Transverse vibration analysis of rotating porous beam with functionally graded microstructure using the differential transform method", *Journal of the Brazilian Society of Mechanical Sciences and Engineering*, vol. 37(4), 2015, p. 1435-1444.
- [7] Eringen A.C., Suhubi E.S., "Nonlinear theory of simple micro-elastic solids-I", *International Journal of Engineering Science*, vol. 2, 1964, p. 189-203.
- [8] Eringen A.C., "On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves", *Journal of Applied Physics*, vol. 54, 1983, p. 4703-4710.
- [9] Fleck N.A., Muller G.M., Ashby M.F., Hutchinson, J.W., "Strain gradient plasticity: theory and experiment", *Acta Metall. Mater.*, vol. 42, 1999, p. 475–487.
- [10] Ghannadpour S.A.M., Mohammadi B., Fazilati J., "Bending, buckling and vibration problems of nonlocal Euler beams using Ritz method", *Composite Structures*, vol. 96, 2013, p. 584–589.
- [11] Hosseini Hashemi S., Bakhshi Khaniki H., "Analytical solution for free vibration of a variable cross-section nonlocal nanobeam", *IJE TRANSACTIONS B Applications*, vol. 29(5), 2016, p. 688-696.
- [12] Khajeansari A., Baradaran G.H., Yvonnet J., "An explicit solution for bending of nanowires lying on Winkler–Pasternak elastic substrate medium based on the Euler–Bernoulli beam theory", *International Journal of Engineering Science*, vol. 52, 2012, p. 115–128.
- [13] Lam D.C.C., Yang F., Chong A.C.M., Wang J., Tong P., "Experiments and theory in strain gradient elasticity", *J. Mech. Phys. Solids*, vol. 51, 2003, p. 1477–1508.
- [14] Liew K.M., He X.Q., Kitipornchai S., "Predicting nanovibration of multi-layered graphene sheets embedded in an elastic matrix", *Acta Materialia*, vol. 54, 2006, p. 4229–4236.
- [15] Malekzadeh P., Shojaei M., "Surface and nonlocal effects on the nonlinear free vibration of non-uniform nanobeams", *Composites Part B*, vol. 52, 2013, p. 84-92.
- [16] Mercan K., Civalek O., "DSC method for buckling analysis of boron nitride nanotube (BNNT) surrounded by an elastic matrix", *Composite Structures*, vol. 143, 2016, p. 300-309.
- [17] Murmu T., Pradhan S.C., "Buckling analysis of a single-walled carbon nano-tube embedded in an elastic medium based

on nonlocal elasticity and Timoshenko beam theory and using DQM”, *Physica E.*, vol. 41, 2009, p. 1232–1239.

[18] Pandeya A., Singhb J., “A variational principle approach for vibration of non-uniform nanocantilever using nonlocal elasticity theory”, *Proced. Mater. Sci.*, vol. 10, 2015, p. 497-506.

[19] Peddieson J., Buchanan G.R., McNitt R.P., “Application of nonlocal continuum models to nanotechnology”, *Int. J. Eng. Sci.*, vol. 41(3-5), 2003, p. 305–312.

[20] Phadikar J., Pradhan S., “Variational formulation and finite element analysis for nonlocal elastic nanobeams and nanoplates”, *Computational Mat Sci*, vol. 49(3), 2010, p. 492-499.

[21] Reddy J.N., “Nonlocal theories for bending, buckling and vibration of beams”, *Int J Eng Sci*, vol. 45, 2007, p. 288–307.

[22] Rahmanian M., Torkaman-Asadi M.A., Firouz-Abadi R.D., Kouchakzadeh M.A., “Free vibrations analysis of carbon nanotubes resting on Winkler foundations based on nonlocal models”, *Physica B*, vol. 484, 2016, p. 83–94.

[23] Refaieinejad V., Rahmani O., Hosseini S.A.H., “An analytical solution for bending, buckling, and free vibration of FG nanobeam lying on Winkler-Pasternak elastic foundation using different nonlocal higher order shear deformation beam theories”, *Scientia Iranica F*, vol. 24(3), 2017, p. 1635-1653.

[24] Stolken, J.S., Evans, A.G., “Microbend test method for measuring the plasticity length scale”, *Acta Materialia* vol. 46, 1998, p. 5109–5115

[25] Sudak L.J., “Column buckling of multi-walled carbon nanotubes using nonlocal continuum mechanics”, *J. Appl. Phys.*, vol. 94(11), 2003, p. 7281–7287.

[26] Torabi K., Rahi M., Afshari H., “Transverse Vibration for Non-uniform Timoshenko Nano-beams”, *Mechanics of advanced Composite Structures*, Vol. 2 2015, p. 1-16.

[27] Tsiatas G.C., “A new efficient method to evaluate exact stiffness and mass matrices of non-uniform beams resting on an elastic foundation”, *Archive of Applied Mechanic*, vol. 84, 2014, p. 615–623.

[28] Wang CM, Zhang YY, He X.Q., “Vibration of Non-local Timoshenko Beams”, *Nanotechnology*, vol. 18, 2007, p. 1–9.

[29] Wang Q., Liew K.M., “Application of nonlocal continuum mechanics to static analysis of micro-and nano-structures”, *Physics Letters A*, vol. 363, 2007, p. 236–242.

[30] Wang B.L., Hoffman M., Yu A.B.M., “Buckling analysis of embedded nanotubes using gradient continuum theory”, *Mech Mater*, vol. 45, 2012, p. 52–60.