Optimized mesh generation by colliding bodies optimization algorithm in finite element

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Abstract:

This article presents combination method of h-refinement and node movement in finite element method to solve elasticity problems. Colliding bodies optimization algorithm (CBO), which is a meta-heuristic algorithm, is used to move nodes and in case of inaccurate answers h-refinement could be used to increase the number of nodes in the regions which have too many mistakes. Error estimate, used in both node movement and h-refinement, is made by L2-norm which is appropriate to triangle elements and another use of it is to build cost function that is used in CBO. The proposed method is suitable for finite element meshing procedure because it can solve problems in areas with high stress concentration. Two benchmark example results in linear elasticity problems with respect to other techniques, show the efficiency and acceptable accuracy of the proposed method.

1. Introduction

The development of adaptive finite element method relies on two important techniques, namely the adaptive mesh refinement strategy and the error estimator. The adaptive finite element strategy can be broadly classified into three categories, h-refinement, p-refinement and r-refinement. The h-refinement is to refine the meshes in the region where the error is relatively large and coarsen the meshes in the region where the error is relatively small. The p-refinement, instead, increases the order of the polynomial functions instead of directly refining the mesh. The r-refinement is to move the nodes to increase the mesh density in the region of interest without changing the number of nodes or cells present in a mesh or changing the connectivity of a mesh (Zienkiewicz, 2000[15] & Zienkiewicz, 2006[16]).

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Some hybrid refinement strategies such as hp-type and hr-type refinements used in this paper ae also developed. There are two types of error estimators, the priori error estimator and posteriori error estimator. The priori error estimator provides the error estimate by using well-chosen approximation of the exact solution but not an actual error estimate for a given mesh. In contrast, the posteriori error estimator uses the approximate solution itself to construct the error estimate which can be directly computed based on the approximate solution on a given mesh. Zienkiewicz et all 1987[17] proposed a posteriori error estimator to estimate error according to recovery stresses. Another type of error estimation method is L2 norm, suitable for triangular elements, which is also used in this paper (Johnson 1987[4]; Johnson et al 1991[5]). It is an efficient method that is completely presented in the second part of this article. Colliding bodies optimization (CBO) which belongs to family of meta-heuristic algorithms was recently developed by the authors (Kaveh et al, 2015[6]; Kaveh et al, 2014[7]). This algorithm can be considered as a multi-agent method, where each agent is a colliding body (CB). Simple formation and no internal parameter tuning are advantages of this

algorithm. Before this, the same algorithm which is called charged system search algorithm represented by (Arzani and Kaveh, 2014[2]) in order to reduce errors in meshless procedure was used. In this procedure, they used the method of discrete least squares to determine the error. The method of discrete least squares was primarily used to solve Poisson's equation and then spread to other areas (Arzani and Afshar, 2006[1]).

In the second section of this paper the estimator of error and in the third section, optimization algorithm is presented. In the fourth section, examples of elastic problems, for which the analytical solutions exist, and comparisons between the results of the proposed method and the methods of other researchers are provided.

2. Error estimation and domain refinement in finite element method

2.1. Error estimation in finite element method

Equations occupying the domain of a standard problem can be simply expressed in elliptic-parabolic space as a subset of partial differential equations (Johnson 1987[4]):

$$\nabla u (x) = f (x), \qquad x \in \Omega,$$
$$u(x) = 0, \qquad\qquad x \in \partial\Omega, \qquad\qquad (1)$$

Where Ω is the domain on R^2 with the boundary of $\partial\Omega$, $R^+ = (0, \infty), \nabla = (\partial^2 u / \partial x_1^2) + (\partial^2 u / \partial x_2^2)$ and functions of f and u_0 are boundary conditions of the problem. The error refers to the difference between finite element problem solution and improved problem solution. In other words, the error of displacement solution is calculated by:

$$e_u = u - u_h \qquad\qquad (2)$$

Where u and u_h refer to finite element problem solution and improved problem solution, respectively. The error estimator function used in this method is based on (Johnson 1987[4]; Johnson et al 1991[5]). In equation (1), the error of finite element linear approximation u_h is calculated by L2 norm as follow:

$$\left\| \nabla(u - u_h) \right\| \le \alpha \left\| hf \right\| + \beta D_h(u_h) \qquad (3)$$

The variables α, β and h refer to geometrical conditions of elements of the domain whose calculations are completely described in the relevant reference. Variable D_h referring to variation rate of the quantity along the edge of element is expressed as follows:

$$D_h (v) = (\sum_{\tau \in E_i} h_\tau^2 [\frac{\partial v}{\partial n_\tau}]^2)^{1/2} \qquad (4)$$

Where h_τ is length of the edge τ and n_τ are outward unit normal vectors to the edge and the expression given in brackets is the variation rate of the quantity along the edge of element. The resultant value for each three edges of element E_i is then added up. Considering plane stress conditions and equations occupying the domain, error

estimator function of an element in elliptic equation is defined as:

$$E(K) = \alpha \left\| h(f - \alpha u) \right\|_K + \beta(\frac{1}{2} \sum_{\tau \in k} h_\tau^2 (n_\tau . c \nabla u_h)^2)^{1/2} \qquad (5)$$

Where E(K) refers to numerical error of L2 norm for K^{th} element. Consequently, L2 norm error is determined for all elements. The error is employed in both h- and p-refinement. H-refinement uses element errors while p-refinement employs nodal error that has been obtained from interpolation of element error.

2.2. H-refinement of the mesh

To achieve desirable results, when the domain is discretized, the order of elements is constant in this technique while, the number and size of elements varies. When error of each element is determined, elements producing more than permitted error are detected and selected for refinement in the next step. Adaptive refinement and re-meshing can be generally used for this purpose. In adaptive refinement, location of existing nodes is preserved in each step and some new nodes are introduced in elements and added to the domain. In each step of re-meshing, all existing elements are initially removed and the domain is re-discretized by more nodes focused on regions with higher error. Adaptive refinement is applied in this study, in which bisecting the longest side is used to improve the mesh and create new elements. Based on triangular elements, bisecting the longest side was first introduced in Rosenberg et al 1975[11] to generate new meshes. In recent years, this method has been frequently used due to its simplicity and efficiency (Plaza et al 2005[10]; Yershov et al 2016[13]).

3. Colliding bodies optimization algorithm

3.1. Introduction of algorithm

The colliding bodies optimization is based on momentum and energy conservation law for one dimensional collision (Kaveh et al, 2014[7]). This algorithm contains a number of colliding bodies(CB) where each one is treated as an object with specified mass and velocity that collide with others. After collision, each CB moves to a new position with new velocity with respect to old velocities, masses and coefficient of restitution. CBO starts with a set of agents determined with random initialization of a population of individuals in the search space. Then, CBs are sorted in an ascending order based on the value of cost function. The sorted CBs are divided equally into two groups. The first group is stationary and consists of good agents. This set of CBs is stationary and their velocity before collision is zero. The second group consists of moving agents which move toward the first group. Then, the better and worse CBs, i.e. agent with upper fitness value, of each group collide together to improve the position of moving CBs and to push stationary CBs towards better positions. The change of the body position represents the velocity of the CBs before collision as:

$$v_i = 0 \quad , \quad i=1,2,...,n \tag{6}$$
$$v_i = x_i - x_{i-n} \quad , \quad i=1,2,...,2n \tag{7}$$

Where, V_i and x_i are the velocity vector and position vector of the ith CB, respectively. 2n is the number of population size. After the collision, the velocity of bodies in each group is evaluated using momentum and energy conservation law and the velocities before collision. The velocity of the CBs after the collision is:

$$v_i' = \begin{cases} \dfrac{(m_{i+n} + \varepsilon m_{i+n})v_{i+n}}{m_i + m_{i+n}} \,,\, i = 1,...,n \\[4mm] \dfrac{(m_i - \varepsilon m_{i-n})v_i}{m_i + m_{i-n}} \,,\, i = n+1,...,2n \end{cases} \tag{8}$$

Where, v_i and v_i' are the velocities of the ith CB before and after the collision, respectively, and the mass of the ith CB is defined as:

$$m_k = \frac{\dfrac{1}{fit(k)}}{\sum_{i=1}^{n} \dfrac{1}{fit(i)}} \,,\quad k = 1,2,...,2n \tag{9}$$

Where fit(i) represents the objective function value of the ith agent. Obviously a CB with good value exerts a larger mass and fewer moves than the bad ones. Also, for maximizing the objective function, the term 1/fit(i) is replaced by fit(i). ε is the coefficient of restitution (COR) and is defined as the ratio of the separation velocity of the two agents after collision to approach velocity of two agents before collision. In this algorithm, this index is defined to control the exploration rates. For this purpose, the COR decreased linearly from unit value to zero. Here, ε is defined as:

$$\varepsilon = 1 - \frac{iter}{iter_{max}} \tag{10}$$

Where $iter$ is the actual iteration number, and $iter_{max}$ is the maximum number of iteration. Here, COR is equal to unity and zero representing the global and local search, respectively. In this way a good balance between the global and local search is achieved by increasing the iteration. The new position of CBs is evaluated using the generated velocities after the collision in the position of stationary CBs:

$$x_i^{new} = \begin{cases} x_i + rand.^* v_i' \quad ,i = 1,...,n \\ x_{i-n} + rand.^* v_i' \quad ,i = n+1,...,2n \end{cases} \tag{11}$$

Where, x_i^{new} and v_i' are the new positions and the velocity after the collision of the ith CB, respectively.

3.2. Objective function

Previously, the same algorithm which is called the charged system search algorithm was used by (Arzani and Kaveh, 2014[2]) in order to refine solving in the domain of elasticity problems in meshless method. In the mentioned research, discrete least squares meshless method was used to solve the equations of the theory of elasticity and equivalent of the error of each node as corresponding electric charge in

the CSS method (Arzani and Afshar, 2006[1]). In that method, the movement of points was performed using colliding bodies' optimization, and finally, the acceptable results were presented for problems. This article seeks to use the colliding bodies optimization algorithm in the standard finite element method with a similar approach. According to the proposed method to estimate the error in the second part, it can be seen that the problem domain error which was found using L2 norm method, has the function of locating the position of nodes forming the finite element network. In this estimate using the colliding bodies optimization algorithm, the nodes forming network were replaced to achieve uniform error and better approximation. When the error is specified for each element using equation (5), the total error of the domains is calculated using sum of element values of error. In this step, the total error of L2 norm is initially normalized for strain energy occupying the domain, which is the cost function of this technique, and if the normalized value exceeds permeable error, the mesh must be moved or refined through CBO algorithm and h-refinement. In the flowchart of Fig.1, the process of achieving a better approximation is displayed.

$$Minimize \quad \left(\frac{\sum_{i=1}^{n} E_i}{\sum_{i=1}^{n} E_i + \sum_{i=1}^{n} U_i} \right) \tag{12}$$

Where E_i and U_i refers to the L2 error and strain energy of the element i and n represents number of elements.
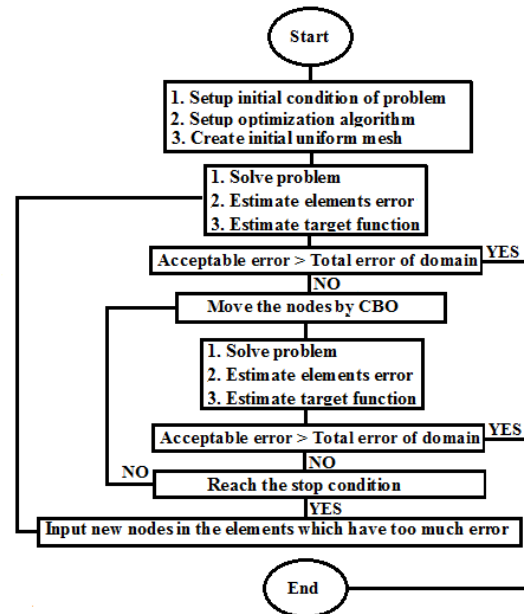

Fig.1: Flowchart of the use of optimization algorithm in finite element

Finally, it is to be reminded that, in the process of solving problems by the proposed method, the proposed mesh is also being examined for mesh quality. This control is precluded to prevent the production of low-quality elements that prevent good accuracy results. The triangle quality is given by the below formula

$$q = \frac{4a\sqrt{3}}{h_1^2 + h_2^2 + h_3^2} \tag{13}$$

Where "a" is the area and h_1, h_2, and h_3 the side lengths of the triangle and If q > 0.6 the triangle is of acceptable quality and q = 1 when $h_1 = h_2 = h_3$ (Bank, 1990[3]). And in the case of q smaller than 0.6, a new location for the node will be considered.

4. Numerical examples

In this section, two two-dimensional benchmark examples, which were presented (Timoshenko et al, 1970[12]), were solved by the method proposed in this article and their results were compared with analytical results and results of other researchers. The first example is a cantilever beam under load at the free end and the second, an infinite plate with circular hole and loads of uniaxial tension on both sides. Other characteristics for each example have been mentioned in the relevant section.

4.1 Cantilever beam with a parabolic load at the end

In this case, a cantilever beam under the effect of a force with parabolic distribution at the end of the beam was placed and boundary conditions of the example are shown in Fig.2. The analytical solutions to this problem have been introduced by (Timoshenko et al, 1970[12]), which were presented in eqs. (14) to (19).
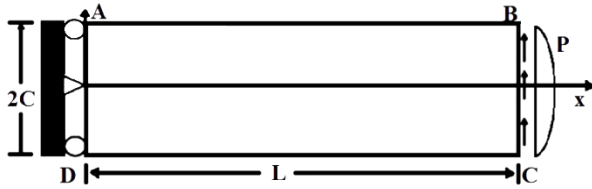
Fig.2: Cantilever beam affected by the parabolic force in free end

$$u = \frac{py}{6EI}\left[3x\left(2L-x\right)+(2+\nu)(y^2-c^2)\right] \qquad (14)$$

$$v = \frac{py}{6EI}\left[x^2\left(3L-x\right)+3\nu\left(L-x\right)y^2+(4-5\nu)c^2x\right] \qquad (15)$$

$$\sigma_x = -p(L-x)y/I \qquad (16)$$

$$\sigma_y = 0 \qquad (17)$$

$$\tau_{xy} = \tau_{yx} = p(c^2-y^2)/2I \qquad (18)$$

$$I = 2c^3/3 \qquad (19)$$

Where

E: the modulus of elasticity,

ν : Poisson coefficient,

In the right border, concentrated load with parabolic distribution has been included and in side of x stress equal to zero is the boundary condition that is applied. Both the upper and lower boundaries are without stress and on the left border, boundary condition of changing location has been considered using analytical solutions. This has been solved by the plane stress conditions and by assuming E=1000 (Pa), ν=0.3, L=12(m), C=1(m) and P=1(N). Numerical solving of the proposed model for different degrees of freedom in this example in Fig.3 and Fig.4, which is related to displacement and stress at AB side was presented. Each line in this figure represents the best response at a certain degree of freedom after the moving of nodes. In other words, initially nodes that

were moved could not reach the permitted error, therefore, the elements with high error are chosen for h-refinement. This cycle is repeated until the appropriate answer is reached.
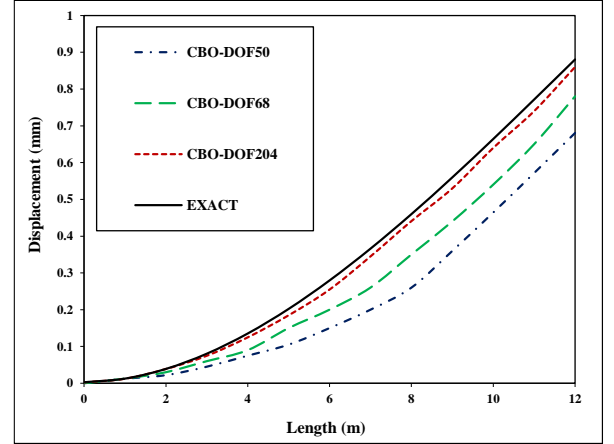

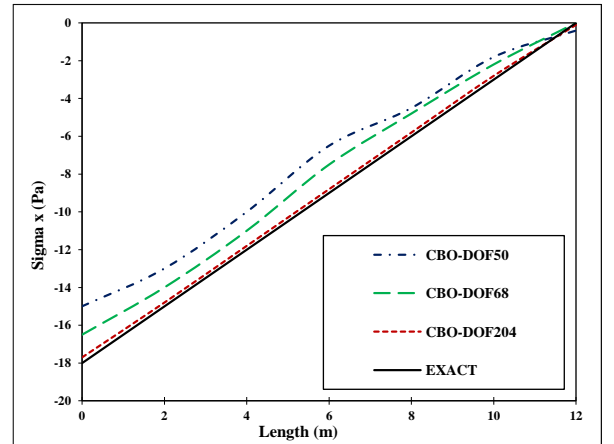**Fig.3:** The results related to the vertical displacement from AB edge

Fig.4: The results related to the vertical stress along of x in AB edge

In order to compare the results of the proposed method with other researchers, beam-profile L=2.4(m), C=0.3(m), P=5000(N), E=3×10^7 (Pa) and ν=0.3 is modelled. In Fig.6, the results of proposed method with the results presented in (Zeng et al., 2016[14]), including the method of NS-FEM (node based smoothed) and ES-FEM (edge based smoothed) and βFEM method and exact analytical method were compared. The method of βFEM is the most recent one to smooth finite element using triangular elements and principles of this method are completely presented in (Zeng et al., 2016[14]). In Fig.5 meshing for DOF-50, DOF-68 and DOF-204 by the proposed algorithm to improve the network related to the cantilever beam is shown.

The result of mesh quality generated at the last stage is presented in Fig.7 where the lowest and highest values of q for the elements produced are respectively 0.84 and 0.9.
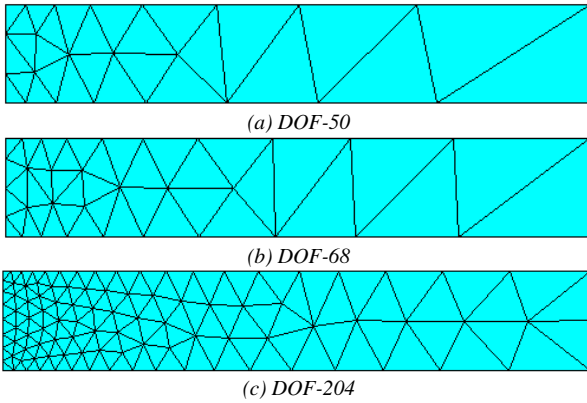
(a) DOF-50

(b) DOF-68

(c) DOF-204

Fig.5: Meshing steps for the different degrees of freedom
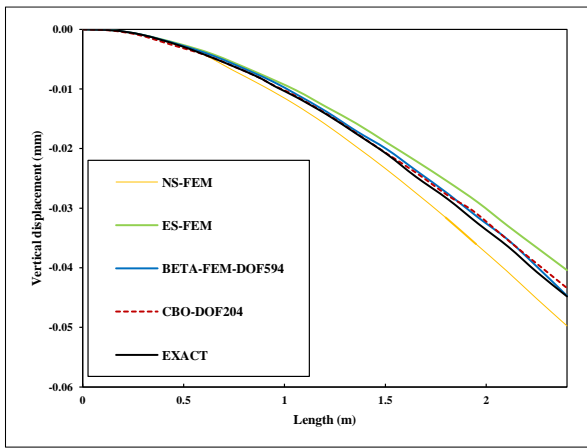

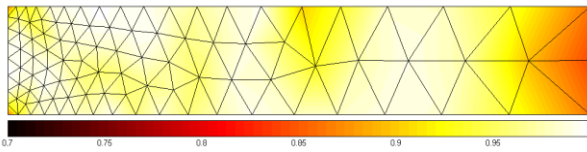**Fig.6:** The results related to the vertical displacement from AB edge

Fig.7: mesh quality for beam example in last step

4.2 Infinite plate with circular hole

In this example, an infinite plate with circular hole according to the Fig.8 that is at the influence of the uniaxial tension force Phase, have been solved. Due to being an infinite plate and according to the symmetry conditions, a quarter of area with 5a size was solved. Boundary and force conditions in Fig.9 are shown. For the boundaries of AB and ED; condition of symmetry, for AE edge; physical condition of free load and for edge of BC and CD; stress boundary condition obtained from the exact solution of problem have been used. The analytical solutions to this problem by (Timoshenko et al, 1970[12]) have been provided and were presented in Eqs. (20) to (25).
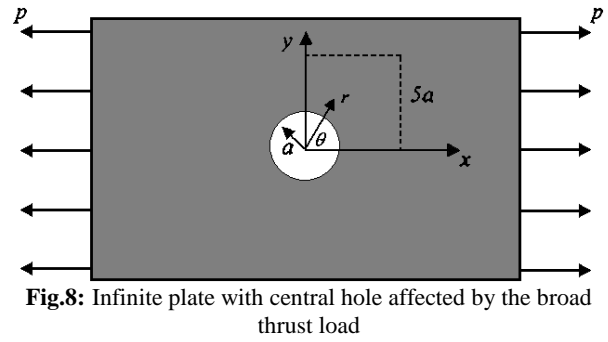

Fig.8: Infinite plate with central hole affected by the broad thrust load
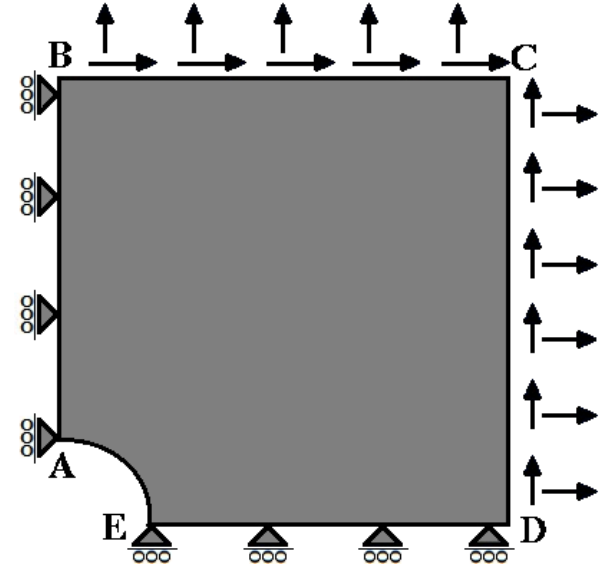

Fig.9: Boundary conditions on the infinite plate or circular hole

$$u_r = \frac{t}{4G}\left[r\left(\frac{k-1}{2} + \cos(2\theta)\right) + \frac{c^2}{r} \times \right.$$

$$\left. \left(1 + (1+k)\cos(2\theta) - \frac{c^4}{r^3}\cos(2\theta)\right) \right] \tag{20}$$

$$u_\theta = \frac{t}{4G}\sin(2\theta)\left[(1-k)\frac{c^2}{r^2} - r - \frac{c^4}{r^3}\right] \tag{21}$$

$$\sigma_x = t\left[1 - \frac{c^2}{r^2}\left(\frac{3}{2}\cos(2\theta) + \cos(4\theta)\right) + \frac{3c^4}{2r^4}\cos(4\theta)\right] \tag{22}$$

$$\sigma_y = -\left[\frac{c^2}{r^2}\left(\frac{1}{2}\cos(2\theta) - \cos(4\theta)\right) + \frac{3c^4}{2r^4}\cos(4\theta)\right] \tag{23}$$

$$\tau_{xy} = -t\left[\frac{c^2}{r^2}\left(\frac{1}{2}\sin(2\theta) - \sin(4\theta)\right) - \frac{3c^4}{2r^4}\sin(4\theta)\right] \tag{24}$$

$$k = (3-\nu)/(1+\nu) \tag{25}$$

Where
υ: Poisson coefficient,
G: Shear modulus.

In this example a=1(m), E=1000(Pa), υ=0.3 and P=1(Pa) are assumed. In Fig.10, meshing for DOF-66, DOF-112 and DOF-290 by the proposed algorithm to improve the network related to the plate with circular hole is shown. Answers obtained for the horizontal displacement of the ED edge of this problem in Fig.11 and normal stress in the X direction for the edge AB in Fig.12 were presented.
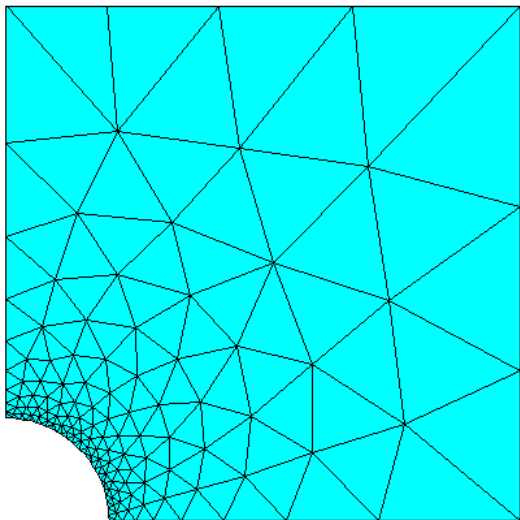
The result of mesh quality generated at the last stage is presented in Fig.13 where the lowest and highest values of q for the elements produced are respectively 0.75 and 1.
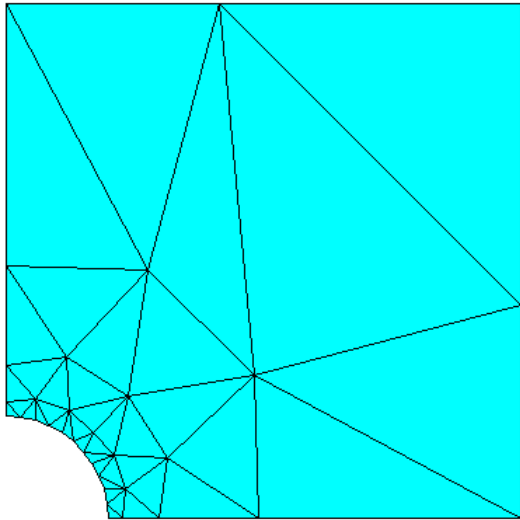
(a) DOF 66
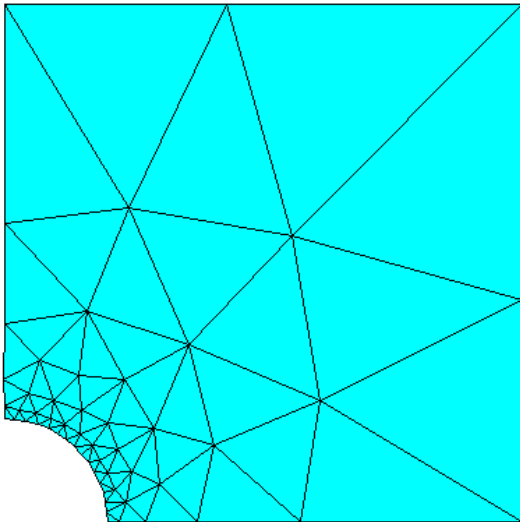
(b) DOF 112

(c) DOF 290
Fig.10: Meshing steps for different degrees of freedom
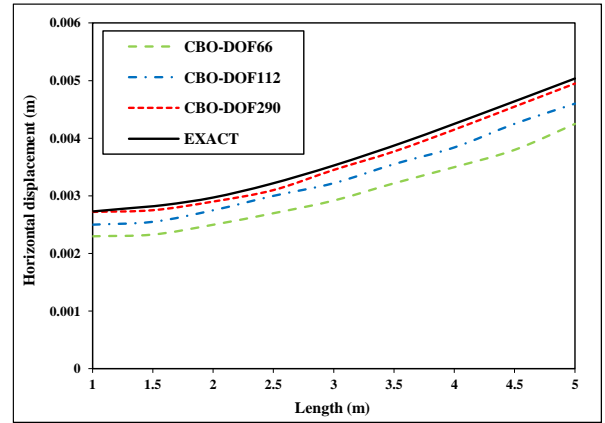

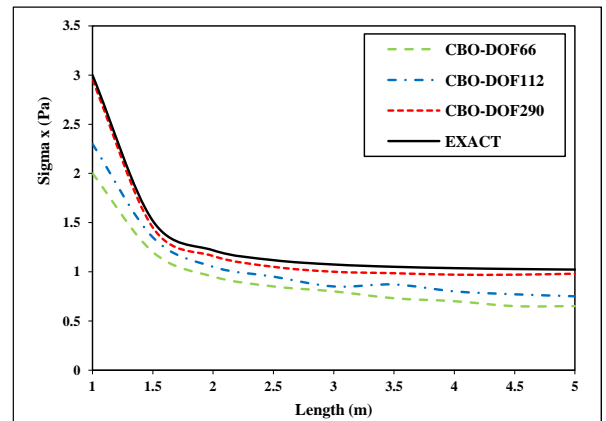**Fig.11:** The results related to the horizontal displacement in ED edge

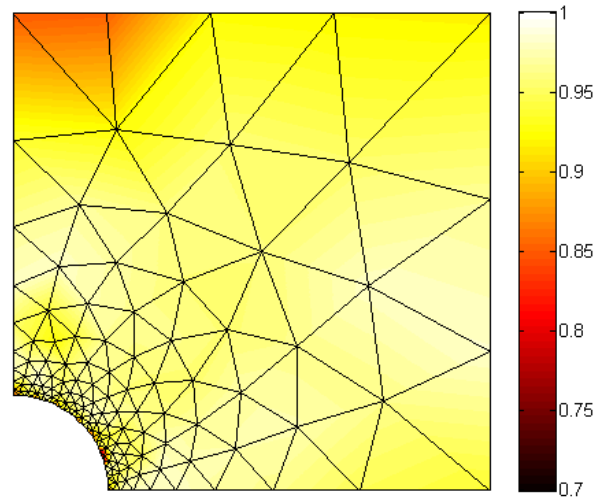**Fig.12:** The results related to the vertical tension in AB edge

Fig.13: mesh quality for plate example in last step

In Fig.14, the results of the horizontal displacement on the edge ED of infinite plate with the circular hole, which has the characteristics of c=1(m) and P=1(Pa), was compared with the method introduced in this article and method presented in reference (Nguyen et al, 2010[6]), including the methods of Aα-FEM, ES-FEM and NS-FEM and exact analytical solutions. In Fig.15, comparing the results of the vertical stress in X direction for AB edge between the method proposed in this article and Aα-FEM method and exact analytical solutions, are displayed. The

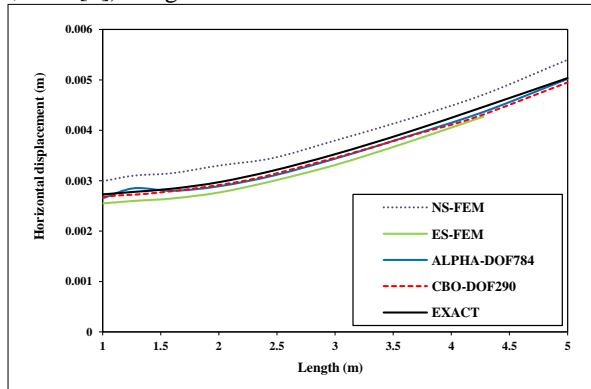principles and laws on the Aα-FEM method in (Nguyen et al, 2010[6]) are given.

Fig.14: The horizontal displacement on the ED edge
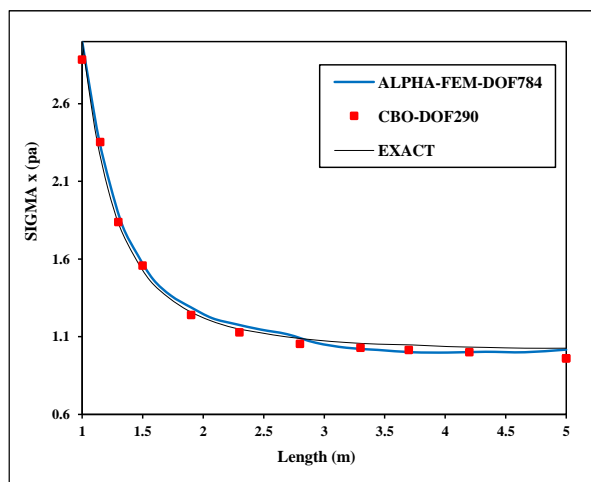

Fig.15: The vertical tension on the AB edge

5. Conclusion

In this paper, the combination of optimization algorithm of CBO in the standard finite element was aimed to decrease the L2 norm of error on the solving problem domain firstly by displacement of nodes, and secondly, by increasing their number (if required). The Meta-heuristic algorithm used by displacement of the position of the nodes within the allowable domain, drives the results towards achieving the least and uniform error on the domain. This displacement process of nodes with CBO is able to compensate the good position for a certain number of elements, and when there is a high level of error on domain, it can increase the number of elements and nodes and then re-move the points. In this manner, the cycle continues to achieve suitable results. The major advantage of using this method is that it is intelligent and will continue to regulate operations to achieve an acceptable answer. The application of movement nodes, before increasing the number of nodes by CBO, is good practice to achieve acceptable answers and prevent the increase in the volume of calculations. According to the numerical results of two examples presented and comparing their results with two recent studies, the effectiveness of the proposed approach is evident. Comparing the results shows that the precision required in the proposed method provides at least half of the conventional degrees of freedom in two examples presented.

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