Cover interpolation functions and h-enrichment in finite element method

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Abstract:

This paper presents a method to improve the generation of meshes and the accuracy of numerical solutions of elasticity problems, in which two techniques of h-refinement and enrichment are used by interpolation cover functions. Initially, regions which possess desired accuracy are detected. Mesh improvment is done through h-refinement for the elements existing in those regions. Total error of the domain is thus reduced and limited to the allowable range. In order to increase the accuracy of solutions to an excellent level, the results of mesh refinement are reassessed in the next steps and the nodes exceeding the value of allowable error are determined. The method automatically improves the subdomain by increasing the order of interpolation cover functions which yields to solutions of appropriate accuracy. A comparison of solutions achieved by the proposed method with that of other methods and also the accurate solutions for linear elasticity examples proves acceptable efficiency and accuracy of the proposed method. In this research, we illustrate the power of the strategy through the solutions obtained for various problems.

1. Introduction

Today, numerical methods are known to be effective solutions for analysis of scientific problems. The most popular numerical methods include finite difference method, finite volume method, finite point method and finite element method. The standard finite element method runs into difficulty with highly curved boundaries and lacks enough accuracy (Bathe 2006[3]). In this method, another challenge is to generate finite element mesh by proper number, type and orders of elements.

Aimed at reducing computational costs of discretization and increased solution accuracy, selection of suitable solution is of great importance in this method. Refinement of standard finite element method is thus observed by researchers (Zienkiewicz 2000[22]).

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In h-refinement, domain of the problem is initially explored by error estimator and regions with discontinuous results are then detected and relevant improvement is performed in accordance with the rate of error. Relevant approaches of error estimation are based on energy norm error and L2 norm error. The approaches can estimate error of the whole domain and each element. Some posteriori estimators also known as anticipatory improvement algorithms are merely used for approximation of ordinary differential equations at starting point. In finite element method, posteriori estimators were used for elliptic differential equations with boundary conditions by Babuška & Zienkiewicz 1986[2].

Based on stress recovery rules, Zienkiewicz and Zhu similarly presented a simple technique for total and local error estimation in finite element solutions and a simple form of adaptive analysis by increasing the number of elements (Zienkiewicz & Zhu 1987[23]). Subsequently, they introduced SPR method for error estimation of finite element solutions and adaptive analysis through two articles

(Zienkiewicz & Zhu 1992[24]; Zienkiewicz & Zhu 1992[25]). All the approaches seek for local improvement of the domain being studied. In other words, instead of the whole domain, elements are used only in subdomains whose solutions include high discontinuity. To develop finite element method, Shi 1985[14] proposed manifold numerical method (MNM) as a combination of standard finite element method and DDA method (discontinuous displacement analysis. The formulation of this method for two and three dimensional domain and problems with discontinuous deformation in references (Grayeli & Mortazavi 2006[5]; Liu et al 2004[11]) are presented. This method is developed to solve crack problems and crack propagation within the domain (Xinmei et al 2010[15]; Zhang et al 2010[21]). Manifold numerical method is an effective numerical technique for solving solid mechanics problems in which boundary conditions and numerical integrations are easily applied. In contrast to standard finite element method, this approach suffers a rank deficient global matrix. Therefore, various techniques have been presented to solve that problem (Ghasemzadeh et al 2014[4]). Inspired by numerical manifold method, enrichment by interpolation cover is one of the newest approaches introduced by Bathe et al 2013[9]. To increase convergence rate, the domain was discretised by low-order finite element in their proposed method and its results for 3-node triangular elements and 4-node tetrahedral elements are presented in references (Bathe et al 2013[9]; Bathe et al 2014[10]) via 2D and 3D analyses. The method is based on finite elements enriched by interpolation cover functions on each element which leads to an increased convergence rate of solutions. Furthermore, the method can also be used in distorted elements and the theory used in this method is available in references (Jeon et al 2014[6]; Bathe et al 2013[9]; Bathe et al 2014[10]). Arzani et al 2014[1] and Zeng et al 2016[20] present other attempts to increase accuracy of the results in finite element and meshless method.

Element error estimator and error estimation of the domain is addressed in the second part of this paper. Adaptive analysis of finite element method, interpolation cover functions, the algorithm of proposed method and some elasticity examples are described in the third, fourth, fifth and sixth parts, respectively. Finally, a comparison among proposed method and other approaches adapted by researchers is presented.

2. Error estimation in finite element method

Equations occupying the domain of a standard problem can be simply expressed in elliptic-parabolic space as a subset of partial differential equations (Johnson 1987[7]):

$$\nabla u\,(x) = f\,(x), \quad x \in \Omega,$$
$$u(x) = 0, \qquad\quad x \in \partial\Omega, \tag{1}$$

Where Ω is the domain on R^2 with the boundary of $\partial\Omega$, $R^+ = (0, \infty), \nabla = (\partial^2 u\,/\,\partial x_1^2) + (\partial^2 u\,/\,\partial x_2^2)$ and functions of f and u_0 are boundary conditions of the problem. The error refers to the difference between finite element problem solution and improved problem solution. In other words, the error of displacement solution is calculated by:

$$e_u = u - u_h \tag{2}$$

Where u and u_h refer to finite element problem solution and improved (approximated) problem solution, respectively. The error estimator function used in this method is based on (Johnson 1987[7]; Johnson et al 1991[8]). In equation (1), the error of finite element linear approximation u_h is calculated by L2 norm as follow:

$$\left\| \nabla(u{-}u_h) \right\| \le \alpha \left\| hf \right\| + \beta\,D_h(\,u_h\,) \tag{3}$$

The variables α, β and h refer to geometrical conditions of elements of the domain whose calculations are completely described in the relevant reference. Variable D_h referring to variation rate of the quantity along the edge of element is expressed as follows:

$$D_h(\,v\,) = \Big(\sum_{\tau \in E_i} h_\tau^2 [\frac{\partial v}{\partial n_\tau}]^2 \Big)^{1/2} \tag{4}$$

Where h_τ is length of the edge τ and n_τ are outward unit normal vectors to the edge and the expression given in brackets is the variation rate of the quantity along the edge of element. The resultant value for each three edges of element E_i is then added up. Considering plane stress conditions and equations occupying the domain, error estimator function of an element in elliptic equation is defined as:

$$E(K) = \alpha \left\| h(f{-}\alpha u) \right\|_K + \beta \Big(\frac{1}{2} \sum_{\tau \in k} h_\tau^2 (n_\tau.c\nabla u_h)^2\Big)^{1/2} \tag{5}$$

Where $E(K)$ refers to numerical error of L2 norm for K *th* element. Consequently, L2 norm error is determined for all elements. The error is employed in both h- and p-refinement. H-refinement uses element errors while p-refinement employs element error initially converted to nodal values by interpolation.

3. Adaptive problem solving in finite element method

Standard finite elements are usually influenced by geometry of the model, which leads to high gradient solutions. The simplest way to eliminate high gradients are fragmenting the mesh or increasing the order of approximation function for the entire domain. However, this will lead to additional computational costs. Thus, it is effective to detect some parts of the domain with high gradients in their solutions where mesh refinement or high order elements can be applied. Therefore, adaptive methods

have been considered by many researchers. Adaptive refinement is a strategy to prevent excessive increase in degree of freedom, number of nodes and element with high order. Generally, there are various techniques for adaptive problem solving in finite element method. Finite element mesh refinement is commonly categorized into two general groups. In the first group, mesh refinement is performed by increasing degree of freedom through addition of new nodes (h-refinement) and in the second group by increasing order of elements (p-refinement). Both techniques are utilized in this paper. Each technique is briefly explained in the following (Yang et al 2016[16]; Yang et al 2016 [17]).

3.1. H-refinement of the mesh

To achieve desirable results, when discretising the domain, the order of elements is kept in this technique while the number and size of elements varies. When error of each element is determined in a finite element mesh, elements producing more than permitted error are detected. These elements are used for refinement in the next step. Adaptive refinement and re-meshing can be generally used for this purpose. In adaptive refinement, location of existing nodes is preserved in each step and some new nodes are introduced in elements and added to the domain. In each step of re-meshing, all existing elements are initially removed and the domain is re-discretized by more nodes focused on regions with higher error. Adaptive refinement is applied in this study, in which bisecting the longest side is used to improve the mesh and create new elements. Based on triangular elements, bisecting the longest side was first introduced in Rosenberg et al 1975[13] to generate new meshes. In recent years, this method has been frequently used due to its simplicity and efficiency (Plaza et al 2005[12]; Yershov et al 2016[18]).

3.2. P-refinement of the mesh

Size and number of elements are constant in this technique and mesh refinement is performed for elements with excessive error merely through increasing order of approximation functions. In this paper, the method proposed in Bathe et al 2013[9] is used to enrich points with excessive error. Equations involving this method (cover enrichment functions) are illustrated in section 4.

4. Enrichment by cover interpolation covers

In this section, formulation of enriched finite elements are briefly explained using cover interpolation functions for finite elements of low orders. If a domain is discretized through meshing by standard elements, accuracy of solutions depends on type and size of elements. In this type of enrichment, a cover subdomain is considered for each node. Each subdomain has an interpolation function of a specified order. These subdomains employ functions of higher orders than the standard condition so that more accurate solutions are achieved. In Figure 1a, function h_i is

a linear interpolator for node i, which equals one in the node i and zero in other points relevant to the node, and subdomain of the node i consists of elements connected to it. Using linear functions for interpolation of subdomains leads to fast computation. The region interpolated for the node i by the linear function h_i is called covered region of the node i and shown by C_i (Fig.1.b). As illustrated in Figure 1.c, covered region of triangular element m with three nodes of i, j and k equals contact region of C_i, C_j and C_k. When covered region of nodes is specified, covered regions must be enriched. Enrichment is done using polynomials with order of p. Interpolated value of unknown u in the node i is demonstrated in equation (6) regarding its covered region.

$$P_i^p[u] = u_i + \begin{bmatrix} \overline{x}_i & \overline{y}_i & \overline{x}_i^2 & \overline{x}_i\overline{y}_i & \overline{y}_i^2 & ... & \overline{y}_i^p \end{bmatrix} a_i \qquad (6)$$

In equation (6), variables ($\overline{x}_i$, $\overline{y}_i$) represent distance from the node i and vector a_i shows additional degrees of freedom in the node i within the covered region C_i. According to the explanations above, cover enrichment approximation of field variable u for an element can be expressed by equation (7).

$$u = \sum_{m=1}^{3} h_i u_i + H_i a_i \qquad (7)$$

Where

$$H_i = h_i \begin{bmatrix} \overline{x}_i & \overline{y}_i & \overline{x}_i^2 & \overline{x}_i\overline{y}_i & \overline{y}_i^2 & ... & \overline{y}_i^p \end{bmatrix} \qquad (8)$$

Adding up values of equation (7) for existing nodes in an element and integrating equations (7) and (8), equation (6) can be revised as below.

$$u = \sum_{m=1}^{3} h_i P_i^p \qquad (9)$$

Interpolation by $h_i P_i^p$ is used instead of standard interpolation, in which $h_i P_i^p$ includes common values of field variable P_i^p besides degrees of freedom corresponding to cover functions. The difference is that besides interpolated values by standard interpolation, the method can achieve better results due to cover enrichment functions.

An advantage of this method is that order of cover function is increased only in regions of undesirable accuracy and it is not essential to use enrichment functions in regions with desirable accuracy. It should be noted that the order of enrichment function is very important and if it equals zero, no enrichment is carried out and the results would be the same as ones for standard finite element method.
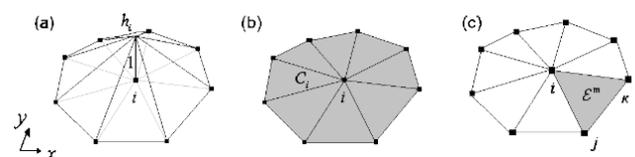
Fig. 1: A description of relationships among enriched subdomains using cover interpolation functions: a) linear interpolation function; b) covered region or elements affected by the by cover interpolation functions; and c) an enriched element.

Performance of proposed method is described here. When the error is specified for each element using equation (5), the total error of the domains is calculated using sum of element values of error. In this step, the total error is initially normalized for strain energy occupying the domain and if the normalized value exceeds 20%, the mesh must be refined through h-refinement.

$$\text{if } \frac{(\sum\limits_{i=1}^{n} e_i)}{n \quad n} < 20 \% \rightarrow \text{stop h-refinement} \tag{10}$$
$$(\sum\limits_{i=1}^{n} e_i + \sum\limits_{i=1}^{n} U_i)$$

Where e_i and U_i refers to the error and strain energy of the element i and n represents number of elements. If equation (10) does not work, h-refinement of the mesh is required. Therefore, the elements in which error exceeds the value determined by equation (11) (50% of maximum element error) are initially detected and employed to generate new elements (bisecting the longest side). When new elements are generated, equation (10) is verified again for the resultant mesh. H-refinement will be halted if the equation validates within the whole domain, otherwise existing elements will be selected again via equation (11) and mesh refinement will continue. The process is iterated until validation of equation (11) is achieved throughout the domain.

$$\text{if } e_i > 0.5 \times \max (e_j) \rightarrow \text{element i}^{\text{th}} \text{ select for h-ref} \tag{11}$$

Where $\max (e_j)$ represents maximum error of element within the domain.

In the next step, the proposed method tends to refine the mesh by cover enrichment functions. Herein the error of all nodes is initially determined through interpolation of element errors and generation of these values to the nodes. According to equation (12), degree of cover enrichment interpolating functions is then determined for each node and used for enrichment as below.

$$\text{if } \begin{cases} n_i \leq 0.3 \times \gamma & \rightarrow p = 0 \\ 0.3 \times \gamma < n_i \leq 0.6 \times \gamma & \rightarrow p = 1 \\ 0.6 \times \gamma < n_i \leq 0.8 \times \gamma & \rightarrow p = 2 \\ n_i > 0.8 \times \gamma & \rightarrow p = 3 \end{cases} \tag{12}$$

Where n_i refers to the error in node i, γ represents 50% of computed maximum nodal error and p refers to order of cover enrichment function for a given node. For clarification, the flowchart of the proposed method is illustrated in Figure 2. An advantage of this method is that enrichment functions are applied only in regions of undesirable accuracy. Cover enrichment functions are used with various orders regarding the accuracy in each node. It prevents an unreasonable increase in computations and enhances the accuracy only in regions with excessive errors.
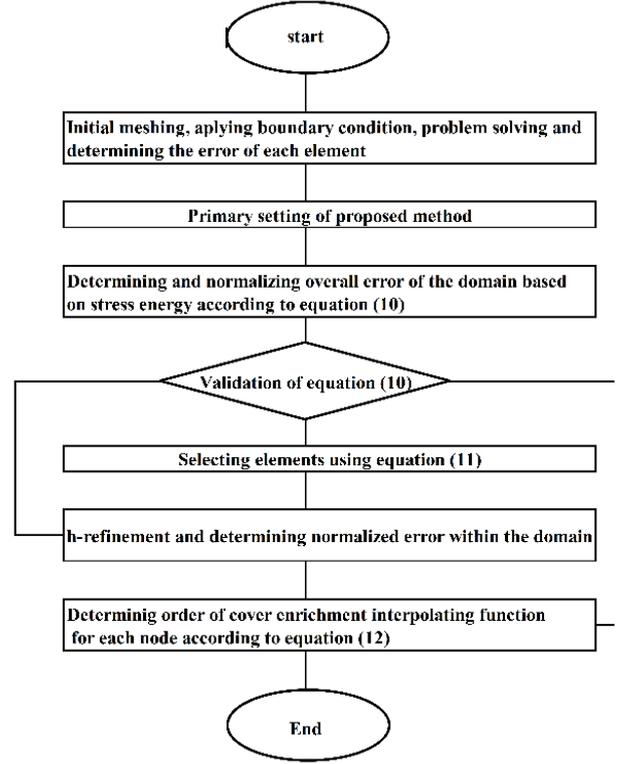
Fig. 2: Flowchart of hp-refinement method.

6. Numerical Examples

Herein two standard 2D elasticity problems are presented with their boundary conditions from Bathe et al 2013[9] and Bathe et al 2014[10] and analyzed by proposed method and the results are compared to other methods adopted by researchers. In the first problem, a cantilever beam subjected to a tip load is investigated. The second problem considers, a compressive distributed load applied to the top side of a cantilever beam with an elliptical hole in the middle.

6.1. The example of a cantilever beam with fillets

In this example, a cantilever beam under plane stress is subjected to a tip load and boundary conditions illustrated in Figure 3 are studied. Since there is no exact solution for the problem, reference Bathe et al 2013[9] has used a very fine mesh including 2460 (9-node) quadrilateral elements to calculate the reference solution.
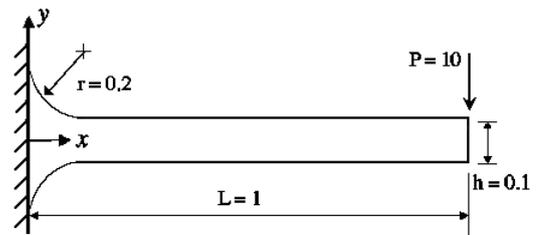
Fig. 3: Conditions of a cantilever beam having curved boundaries

In this problem, elasticity modulus $E=7.2\times 10^9$ Pa, the applied load P=10 KN, poison's ratio υ=0.3 and radius of the curved parts r =0.2m are considered. Figure 4 demonstrates the discretion during problem solving by proposed method. Figure 4.a shows initial meshing with a low degree of freedom for domain discretization at the beginning of problem solving.
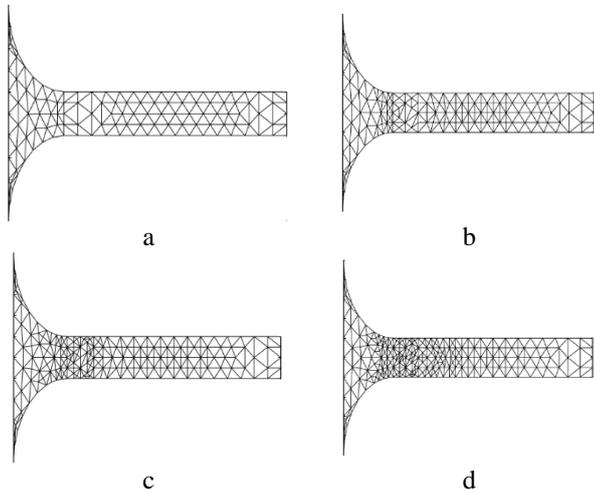
Fig. 4: Meshing during problem solving via proposed method

The results of the problem are illustrated in Figure 5. Values of column 1 is related to the meshing shown in Figure 4.d (last step of h-refinement) and values of column 2 are associated with the meshing shown in Figure 4.d under influence of cover enrichment functions. It must be noted that the values are the best results achieved by proposed method. Displacement in X direction is represented in Figure 5.a and 5.b; displacement in Y direction is represented in Figure 5.c and 5.d; stress distribution is represented in Figure 5.e and 5.f; and stress distribution is represented in Figure 5.g and 5.h.
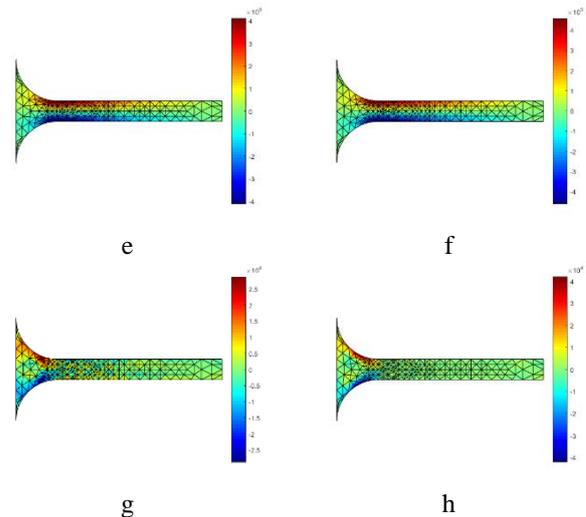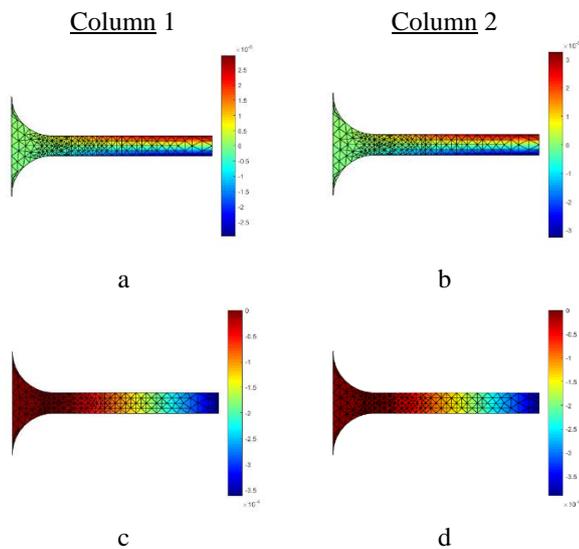
Fig. 5: The results of first problem

To compare proposed method with the other, von Misses stress distribution is represented for the problem in Figure 6. Figure 6.a (2572 DOFs) refers to proposed method in this paper and Figure 6.b (2988 DOFs) is related to the method used in Bathe et al 2013[9].
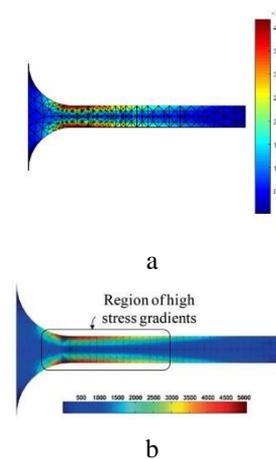
Fig. 6: Comparison of the results based on von Misses stress distribution

6.2. The example of a 2D tool jig problem

In this example, a cantilever beam subjected to a constant pressure load on its top side and boundary conditions illustrated in Figure 7 is assessed. According to Bathe et al 2014[10], the exact solution of the problem has been obtained through a very fine mesh including 40000 (9-node) elements leading to 323200 degree of freedom.
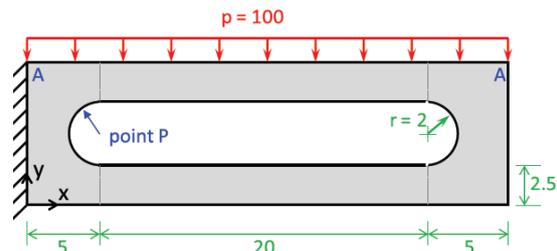
Fig. 7: Conditions of a 2D tool jig problem

In this problem, elasticity modulus E=7.2×10^9 Pa, the applied load P=100 KN/m, poison's ratio υ=0.3 and radius of the curved parts r=2m are considered. Figure 8 demonstrates the meshing during problem solving via proposed method. The problem solving is the same as previous examples. Figure 8.a shows initial meshing and Figure 8.b represents last step of h-refinement.

a

b
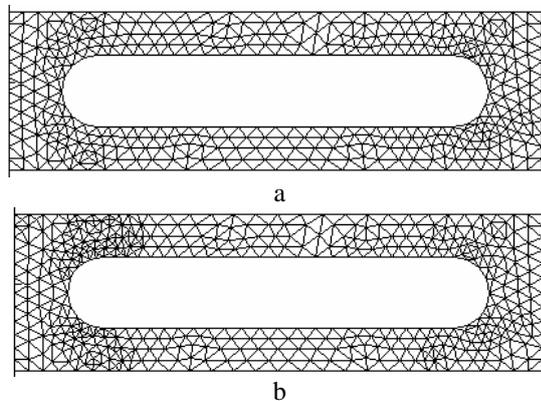
Fig. 8: Meshing during problem solving via proposed method

The results of problem solving are illustrated in Figure 9. Values of column 1 are related to the meshing shown in Figure 8.b (last step of h-refinement) and values of column 2 are associated with the meshing shown in Figure 8.b under influence of cover enrichment interpolating functions. Displacement in X direction is represented in Figure 9.a and 9.b; displacement in Y direction is represented in Figure 9.c and 9.d; stress distribution is represented in Figure 9.e and 9.f; and stress distribution is represented in Figure 9.g and 9.h.

Column 1   Column 2

a   b

c   d

e   f

g   h
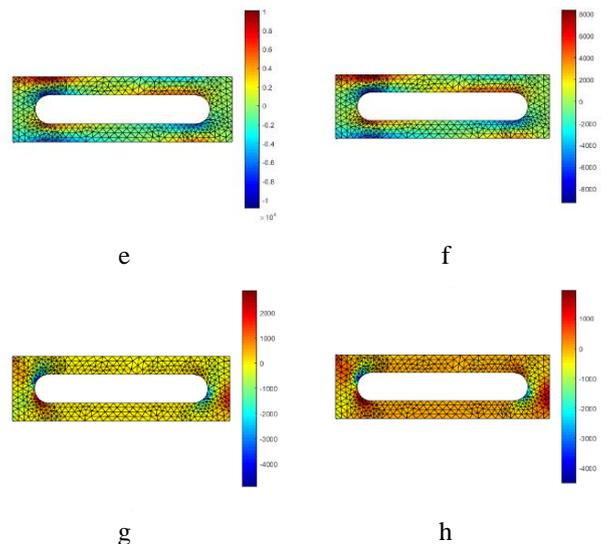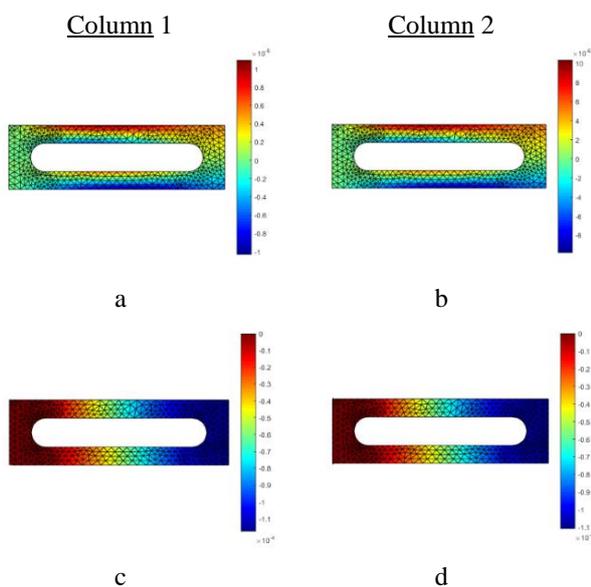
Fig. 9: The results of second problem

To compare proposed method with the other, von Misses stress distribution is represented for the problem in Figure 10. Figure 10.a (2578 DOFs) refers to proposed method in this paper and Figure 10.b (11818 DOFs) is related to the method used in Bathe et al 2014[10].
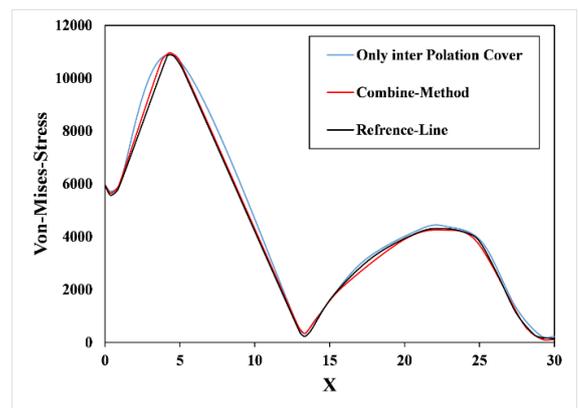
Fig.10: Comparison of the results based on von Misses stress distribution

Figures 9 and 10 demonstrate how adaptive interpolation leads to more accurate results. As seen in Figure 9, the solution is significantly improved using cover interpolations. It should be noted that the adaptive method appropriately distributes cover orders throughout resultant meshing; it mostly employs cover functions of higher orders in case of a coarse mesh while it often utilizes interpolating covers of lower orders in case of a fine mesh. The challenge is properly met using h-refinement in this method. Compared to the method adopted in Bathe et al 2014[10], the results of this example, which are more complex than the first one, use fewer elements and more suitable element density for regions with high gradient. Moreover, quality of the results is greatly improved because of proper distribution of order of enrichment functions in the last step of problem solving.

7. Conclusion

In finite element method (FEM), in order to achieve more accurate results, engineers access no tools for determining proper number and sizes of elements except engineering judgment; sizes of elements are almost selected based on recommendations which possess deficiencies and cannot be applied to all 2D elasticity problems. Uniform fragmentation of the mesh, for instance, is a recommendation never suggested computationally despite proper results. This paper attempts to introduce a combined method to meet the challenge suitably. Consequently, error estimation is initially performed based on L2 norm and the two methods are then employed to refine meshing and increase accuracy of existing solutions. The first method utilizes h-refinement for adaptive mesh improvement. This technique detects regions with excessive error and refines the analysis by bisecting the longest side which leads to increased number of elements within the region. The process continues until validation of error criteria. The second technique employs cover interpolation functions as a powerful tool to eliminate limitations of FEM. According to error of nodes, the order of cover interpolation function is calculated for each node within the domain and used in new problem solving. Nodal coordinates of points do not change during the calculations, but also new points are adaptively introduced within the domain using first technique in regions of less accuracy and when proper meshing is achieved, second technique will lead to more accurate results in problem solving.

Advantages of proposed method include using a standard and efficient norm to determine the error, adaptive mesh generation, its validation for many 2D elasticity problems with extreme complexity within their domains and automatic performance in both steps of h-refinement and determining order of cover enrichment functions. This method can considerably reduce computational attempts and properly enhance accuracy of analytical results. Some examples are provided to indicate performance of the method.

This paper attempts to use fewer elements at the beginning and subsequently introduces an individual generated indicator to determine order of cover enrichment functions. In fact, it aims to suggest a method for automatic refinement in 2D problems. Improvement of stress field and displacement within the domain is evident in the results of previous sections. Efficiency of proposed method is indicated through an investigation into the results of standard examples and other researchers' works. A comparison of the results of this paper with other researchers' works demonstrates efficient accuracy with reasonable degree of freedom in FEM.

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