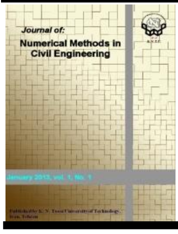

Numerical Methods in Civil Engineering

Vibration characteristics of axially loaded tapered Timoshenko beams made of functionally graded materials by the power series method

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Abstract:
In the present article, a semi-analytical technique to investigate free bending vibration behavior of axially functionally graded non-prismatic Timoshenko beam subjected to a point force at both ends is developed based on the power series expansions. The beam is assumed to be made of linear elastic and isotropic material with constant Poisson ratio. Material properties including the elastic modulus and mass density vary continuously through the beam axis according to the volume fraction of the constituent materials based on exponential and power-law formulations. Based on Timoshenko beam assumption and using small displacements theory, the free vibration behavior is governed by a pair of second order differential equations coupled in terms of the transverse deflection and the angle of rotation due to bending. According to the power series method, the exact fundamental solutions are found by expressing the variable coefficients presented in motion equations including cross-sectional area, moment of inertia, material properties and the displacement components in a polynomial form. The free vibration frequencies are finally determined by solving the eigenvalue problem of the obtained algebraic system. Four comprehensive examples of axially non-homogeneous Timoshenko beams with variable cross-sections are presented to clarify and demonstrate the performance and convergence of the proposed procedure. Moreover, the effects of various parameters like cross-sectional profile and material variations, taper ratios, end conditions and concentrated axial load are evaluated on free vibrational behavior of Timoshenko beam. The obtained outcomes are compared to the results of finite element analysis in terms of ABAQUS software and those of other available numerical and analytical solutions. The competency and efficiency of the method is then remarked.

1. Introduction

Functionally graded materials (FGMs) are advanced multi-phase composites with the volume fraction of particles varying smoothly through the thickness or longitudinal direction of the member. Compared to laminated composites, some advantages of FGM, for instance, elimination of interfacial stress concentration and cracks exist. Introducing functionally graded materials made up of metal and ceramic in recent years have led to some remarkable outcomes such as increasing strength and thermal resistance of mechanical components. Therefore, the use of FG structures has been increasing substantially in a variety of engineering applications including aircrafts, turbine blades, rockets and fuselage structures.

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Beams with variable cross-section are also extensively spread in aeronautical and mechanical components in order to increase stability of structure, satisfy functional requirements and also reduce weight and cost. Due to improvements in fabrication process and combination of the advantages of non-prismatic beam and functionally graded material, the designers can produce structures with favorable strength and manage the distribution of material properties.
The exact estimation of natural frequency and vibration analysis of elastic members under harmonic loads are an essential prerequisite in design of structures. Therefore, different numerical techniques, fundamentally based on the Finite Element Method (FEM), have been proposed for vibration analysis of homogeneous and non-homogeneous Euler-Bernoulli beams with uniform or non-uniform cross-sections. According to Euler-Bernoulli beam theory, the influences of shear deformation and rotatory inertia are negligible and only the effect of flexural deformation is taken into account. Due to these characteristics, the transverse natural frequencies of elastic beam are usually

overestimated based on Euler-Bernoulli model. A large number of researches are dedicated to the free transverse vibration of Euler-Bernoulli beams. Among the early investigations on this topic, the most important one is the study of Karabalis, 1983 [17], in which a finite element model is proposed for dynamic and stability analyses of plane tapered beams with variable depth. A numerical technique based on Galerkin's method was represented by Jategaonkar and Chehil, 1989 [16] to analyze the free vibration behavior of beams with variable geometrical properties. A finite element approach was proposed by Kim and Kim, 2000 [3] to investigate the free vibration behavior of tapered beams with I-section. Elishakoff and Becquet, 2000 [8]; Elishakoff and Guede, 2001 [9]; Elishakoff and Guede, 2004 [10] adopted the semi inverse method to investigate the stability and dynamic characteristics of inhomogeneous beams under different circumstances. Singh and Li, 2009 [28] determined the natural frequencies of a non-uniform beam using a new developed numerical method. Huang and Li, 2010 [13] investigated free vibration of axially FG non-prismatic beam by presenting a new simple approach. In their study, the governing motion equation was transformed to the corresponding Fredholm integral equations. More recently, a numerical finite element method to investigate free vibration behavior of functionally graded beams was developed by Alshorbagy *et al*., 2011 [1], in which the material properties are assumed to vary in the length or thickness directions by a simple power-law. The linear static and dynamic analyses of homogeneous and axially non-homogeneous tapered Euler-Bernoulli beams were performed by Shahba *et al.*, 2011 [24]; Shahba *et al.*, 2013 [26] and Shahba *et al.*, 2013 [27] using a general finite element model. They derived new shape functions in terms of basic displacement functions (BDFs), which are based on energy method.

It should be noted that in the current study, Timoshenko beam theory is adopted for analyzing beams with non-uniform cross-section, in which the influences of shear deformation and rotatory inertia are taken into consideration in the calculation process. Researchers usually apply Timoshenko beam for static and dynamic analyses of elastic members such as towers, moveable arms, thick and ambulatory beams. However, free vibration analysis of Timoshenko beams is complicated because of coupling of slope and flexural displacement. The task seems to be more complex in the presence of axially functionally graded beams where the cross-section properties are not constant.

Irie *et al.*, 1980 [15] adopted the transfer matrix approach for the vibration and stability analyses of Timoshenko beams subjected to a tangential follower force. A Finite Element solution was then proposed by Yokoyama, 1988 [34] to estimate the natural frequencies and the critical buckling loads of Timoshenko beams on Winkler type elastic foundation. Lee and Lin, 1992 [19] investigated the exact free vibration of non-prismatic Timoshenko beams with attachments by using Frobenius method. Based on the step-reduction method, a new approach for the free and forced vibration of non-homogeneous Timoshenko beams with non-uniform cross-sections was developed by Tong and Tabarrok, 1995 [32]. Lueschen *et al.*, 1996 [20] investigated the vibration of uniform Timoshenko beams based on Green's functions. Esmailzadeh and Ohadi, 2000 [11] studied the free vibration and stability analyses of non-prismatic Timoshenko beams subjected to axial and tangential loads by transforming two governing differential equations into one-fourth order differential equation with variable coefficients. The exact solution was finally presented by the method of Frobenius. A new numerical algorithm, based on the differential quadrature element method (DQEM), was presented by Chen, 2002 [7] to study free vibration behavior of non-uniform Timoshenko beams resting on elastic foundation. Auciello and Ercolano, 2002 [5] introduced a new dynamic method to analyze Timoshenko beams with linear variation of depth along their lengths by considering the effects of shear deformation and rotary inertia. In their study, the variational iteration Rayliegh-Ritz method considering orthogonal polynomials as test functions was adopted. Ruta, 2006 [23] used Chebyshev polynomial method to solve the coupled system of equilibrium differential equations of Timoshenko beams with non-uniform cross-section resting on a two parameter elastic foundation. The effects of generalized damping on the non-homogeneous Timoshenko beams were investigated by Sorrentino *et al.*, 2007 [31]. Ozdemir and Kaya, 2008 [21] analyzed free vibration of a rotating, double-tapered Timoshenko beam under flapwise bending vibration by using differential transformation method. They obtained kinetic and potential energy expressions and eventually, obtained the governing differential equations by using Hamilton's principle. A new Finite Element technique for the non-linear free and forced vibration analyses of non-prismatic Timoshenko beams resting on a two parameter foundation was applied by Zhu and Leung, 2009 [35]. Attarnejad *et al.*, 2011 [4] investigated elastic stability analysis of Timoshenko beams with variable cross-section by employing a new finite element approach in which new shape functions in terms of basic displacement functions (BDFs) were derived. A Finite Element solution, based on combination of energy approach and basic displacement functions (BDFs) for free vibration and stability analyses of Functionally Graded (FG) non-prismatic Timoshenko beams, was represented by Shahba *et al.,* 2011 [25]. Moreover, Rajasekaran, 2013 [22] inspected free vibration behavior of axially FG tapered Timoshenko beams using two different techniques: Differential Transformation Method (DTM) and Differential Quadrature Element Method (DQEM). Huang *et al.*, 2013 [14] investigated free vibration of axially FG non-prismatic Timoshenko beam by presenting a new numerical method. In their study, the coupled system of governing differential equations of Timoshenko beam was transformed into a single equation by introducing an auxiliary function.

The main purpose of the current study is calculating the free vibration frequencies for any type of axially functionally graded non-prismatic Timoshenko beams with linear, polynomial or exponential variation of mechanical properties by using an efficient and reliable numerical technique. Based on the Timoshenko beam theory, there exist two coupled motion equations with variable coefficients for deflection and rotation. Investigation of the vibration behavior of non-uniform Timoshenko beams could be complicated due to the presence of shear deformation and variation in geometrical properties over the member's length. The power series approximation is thus adopted to facilitate the solution of system of coupled differential equations with variable coefficients. The mentioned mathematical approach was applied by the author to obtain the critical buckling loads and natural frequencies of elastic members with variable cross-section (Asgarian *et al.*, 2013 [2]; Soltani *et al.*, 2014 [29]; Soltani and Mohri, 2016 [30]). Regarding this numerical methodology, the functions describing the beam's mechanical characteristics such as: flexural and shear rigidities and material properties are expanded into power series form. According to the aforementioned method, explicit expressions for deformation shapes including the transverse deflection and rotation are also identified. The circular frequencies of the considered member are then derived by imposing the boundary conditions and solving the eigenvalue problem.

After presenting the above steps, four numerical examples of non-prismatic Timoshenko beams are performed in order to verify the validity, correctness and competency of the proposed numerical approach with the available outcomes represented in the literature. The numerical results of simply supported, clamped-clamped and cantilever members are presented by considering the effects of axially non-homogenous material and external concentrated axial load. Comments and conclusion close the manuscript.

2. Formulations

An axially functionally graded Timoshenko beam with variable cross-section as depicted in Fig.1 is taken into account. Following the Timoshenko beam theory, differential equations of harmonic motion in the absence of damping and external transverse loadings can be expressed as:

$$(EI\,\theta')' + kGA(w' - \theta) + \rho I\,\omega^2\theta = 0 \tag{1a}$$

$$(kGA(w' - \theta))' - Pw'' + \rho A\omega^2 w = 0 \tag{1b}$$

In the previous equations, w signifies the vertical displacement (in z direction), and θ represents the angle of rotation of the cross-section due to bending. E, G and ρ denote Young's modulus, the shear modulus and the material density, respectively. I and A express the second moment of inertia and cross-sectional area. $\boldsymbol{\omega}$ and $\boldsymbol{k}$ are natural frequency (circular) and the shear correction factor. In the current research, the beam is loaded by a constant axial load (***P***), which is tangential to the x-axis of member. It is also assumed that the point axial load is applied at the end beam without any eccentricities.

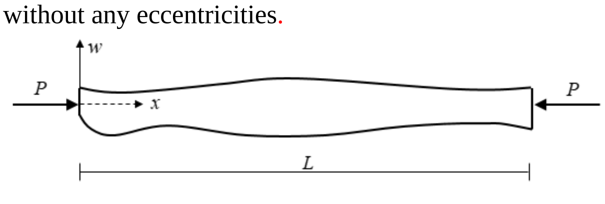

Fig. 1: A non-prismatic Timoshenko beam under axial load

Due to the presence of Timoshenko beam with non-uniform cross-section, the geometrical characteristics of the cross-section over the member length are variable (*I(x), A(x)*). For the axially functionally graded material, the properties of elastic material including the mass density, shear and Young's moduli are also arbitrary along the *x*-axis (*E(x), G(x), ρ(x)*). For these reasons, all the variable terms in Eq. (1) are presented in power series form, as follows:

$$I(x) = \sum_{i=0}^{\infty} I_i x^i, \quad A(x) = \sum_{i=0}^{\infty} A_i x^i, \quad E(x) = \sum_{i=0}^{\infty} E_i x^i$$
$$G(x) = \sum_{i=0}^{\infty} G_i x^i, \quad \rho(x) = \sum_{i=0}^{\infty} \rho_i x^i \tag{2}$$

Where I_i, A_i, E_i, … are coefficients of power series at order i. In order to facilitate the solution of the system of motion equations, a non-dimensional variable ($\varepsilon = x/L$) is introduced and expressions of Eq. (2) are then written in terms of ε as:

$$I(\varepsilon) = \sum_{i=0}^{\infty} I_i L^i \varepsilon^i, \quad A(\varepsilon) = \sum_{i=0}^{\infty} A_i L^i \varepsilon^i, \quad E(\varepsilon) = \sum_{i=0}^{\infty} E_i L^i \varepsilon^i$$
$$G(\varepsilon) = \sum_{i=0}^{\infty} G_i L^i \varepsilon^i, \quad \rho(\varepsilon) = \sum_{i=0}^{\infty} \rho_i L^i \varepsilon^i \tag{3}$$

Substituting Eq. (3) and the non-dimensional variable ε into the motion equations (Eq. (1)) lead to the following formulations:

$$\frac{d}{d\varepsilon}\Big(\big(\sum_{i=0}^{\infty} E_i L^i \varepsilon^i\big)\big(\sum_{j=0}^{\infty} I_j L^j \varepsilon^j\big)\frac{d\theta}{d\varepsilon}\Big) + k\big(\sum_{i=0}^{\infty} G_i L^i \varepsilon^i\big)\big(\sum_{j=0}^{\infty} A_j L^j \varepsilon^j\big)\Big(L\frac{dw}{d\varepsilon} - L^2\theta(\varepsilon)\Big) + \omega^2 L^2\big(\sum_{i=0}^{\infty} \rho_i L^i \varepsilon^i\big)\big(\sum_{j=0}^{\infty} I_j L^j \varepsilon^j\big)\theta(\varepsilon) = 0 \tag{4a}$$

$$\frac{d}{d\varepsilon}\left(k\big(\sum_{i=0}^{\infty} G_i L^i \varepsilon^i\big)\big(\sum_{j=0}^{\infty} A_j L^j \varepsilon^j\big)\Big(\frac{dw}{d\varepsilon} - L\theta(\varepsilon)\Big)\right) - P\frac{d^2 w}{d\varepsilon^2} + \omega^2 L^2\big(\sum_{i=0}^{\infty} \rho_i L^i \varepsilon^i\big)\big(\sum_{j=0}^{\infty} A_j L^j \varepsilon^j\big)w(\varepsilon) = 0 \tag{4b}$$

The general solutions of two displacement parameters ($w(\varepsilon), \theta(\varepsilon)$) are also presented by the following power series of the form:

$$w(\varepsilon) = \sum_{i=0}^{\infty} a_i \varepsilon^i \tag{5a}$$

$$\theta(\varepsilon) = \sum_{i=0}^{\infty} b_i \varepsilon^i \tag{5b}$$

According to Eq. (4), first and second displacement derivatives are required. Utilizing Eq. (5) results in:

$$\frac{d(w,\theta)}{d\varepsilon} = \sum_{i=1}^{\infty} i\,(a_i, b_i)\varepsilon^{i-1} = \sum_{i=0}^{\infty}(i+1)(a_{i+1}, b_{i+1})\varepsilon^i \tag{6a}$$

$$\frac{d^2(w,\theta)}{d\varepsilon^2} = \sum_{i=1}^{\infty} i\,(i-1)(a_i, b_i)\varepsilon^{i-2} = \sum_{i=0}^{\infty}(i+1)(i+2)(a_{i+2}, b_{i+2})\varepsilon^i \tag{6b}$$

Furthermore, introducing new variables:

$$I_i^* = I_i L^i, \quad A_i^* = A_i L^i, \quad E_i^* = E_i L^i, \quad G_i^* = G_i L^i \tag{7}$$

And substituting Eqs. (5)-(7) into Eq. (4), the following expressions are obtained:

$$\frac{d}{d\varepsilon}\left((\sum_{i=0}^{\infty}E_i^*\varepsilon^i)(\sum_{j=0}^{\infty}I_j^*\varepsilon^j)(\sum_{k=0}^{\infty}(k+1)b_{k+1}\varepsilon^k)\right)$$
$$+k(\sum_{i=0}^{\infty}G_i^*\varepsilon^i)(\sum_{j=0}^{\infty}A_j^*\varepsilon^j)(L\sum_{k=0}^{\infty}(k+1)a_{k+1}\varepsilon^k-L^2\sum_{k=0}^{\infty}b_k\varepsilon^k)+ \quad (8a)$$
$$+\omega^2L^2(\sum_{i=0}^{\infty}\rho_i^*\varepsilon^i)(\sum_{j=0}^{\infty}I_j^*\varepsilon^j)(\sum_{k=0}^{\infty}b_k\varepsilon^k)=0$$

$$\frac{d}{d\varepsilon}\left(k(\sum_{i=0}^{\infty}G_i^*\varepsilon^i)(\sum_{j=0}^{\infty}A_j^*\varepsilon^j)(\sum_{k=0}^{\infty}(k+1)a_{k+1}\varepsilon^k-L\sum_{k=0}^{\infty}b_k\varepsilon^k)\right)$$
$$-P\sum_{k=0}^{\infty}b_{k+2}(k+1)(k+2)\varepsilon^k$$
$$+\omega^2L^2(\sum_{i=0}^{\infty}\rho_i^*\varepsilon^i)(\sum_{j=0}^{\infty}A_j^*\varepsilon^j)(\sum_{k=0}^{\infty}a_k\varepsilon^k)=0 \quad (8b)$$

After some simplifications, the following expressions are obtained:

$$\sum_{k=0}^{\infty}\sum_{j=0}^{k+1}\sum_{i=0}^{j}E_i^*I_j^*b_{k-j+2}(k-j+2)(k+1)\varepsilon^k$$
$$+Lk\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{j}G_i^*A_{j-i}^*a_{k-j+1}(k-j+1)\varepsilon^k$$
$$-L^2k\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{j}G_i^*A_{j-i}^*b_{k-j}\varepsilon^k \quad (9a)$$
$$+\omega^2L^2\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{j}\rho_i^*I_{j-i}^*b_{k-j}\varepsilon^k=0$$

$$k\sum_{k=0}^{\infty}\sum_{j=0}^{k+1}\sum_{i=0}^{j}G_i^*A_j^*a_{k-j+2}(k-j+2)(k+1)\varepsilon^k$$
$$-Lk\sum_{k=0}^{\infty}\sum_{j=0}^{k+1}\sum_{i=0}^{j}G_i^*A_j^*b_{k-j+1}(k+1)\varepsilon^k$$
$$-P\sum_{k=0}^{\infty}b_{k+2}(k+1)(k+2)\varepsilon^k \quad (9b)$$
$$+\omega^2L^2\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{j}\rho_i^*A_{j-i}^*a_{k-j}\varepsilon^k=0$$

Or

$$\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k+1}\sum_{i=0}^{j}E_i^*I_j^*b_{k-j+2}(k-j+2)(k+1)\right.$$
$$+Lk\sum_{j=0}^{k}\sum_{i=0}^{j}G_i^*A_{j-i}^*a_{k-j+1}(k-j+1)-L^2k\sum_{j=0}^{k}\sum_{i=0}^{j}G_i^*A_{j-i}^*b_{k-j}$$
$$\left.+\omega^2L^2\sum_{j=0}^{k}\sum_{i=0}^{j}\rho_i^*I_{j-i}^*b_{k-j}\right)\varepsilon^k=0 \quad (10a)$$

$$\sum_{k=0}^{\infty}\left(k\sum_{j=0}^{k+1}\sum_{i=0}^{j}G_i^*A_j^*a_{k-j+2}(k-j+2)(k+1)\right.$$
$$-Lk\sum_{j=0}^{k+1}\sum_{i=0}^{j}G_i^*A_j^*b_{k-j+1}(k+1)-Pb_{k+2}(k+1)(k+2) \quad (10b)$$
$$\left.+\omega^2L^2\sum_{j=0}^{k}\sum_{i=0}^{j}\rho_i^*A_{j-i}^*a_{k-j}\right)\varepsilon^k=0$$

According to the above formulations and from mathematical point of view, it is culminated that all the a_i and b_i coefficients except for the first two (a_0, a_1 *and* b_0, b_1) can be obtained. Therefore, the fundamental solution of the motion equations of functionally graded Timoshenko beam with variable cross-section can be expressed in the following forms:

$$w(\varepsilon)=a_0w_0(\varepsilon)+a_1w_1(\varepsilon)+b_0w_2(\varepsilon)+b_1w_3(\varepsilon) \quad (11a)$$
$$\theta(\varepsilon)=a_0\theta_0(\varepsilon)+a_1\theta_1(\varepsilon)+b_0\theta_2(\varepsilon)+b_1\theta_3(\varepsilon) \quad (11b)$$

All terms of $\theta_i(\varepsilon)$ and $w_i(\varepsilon)$ are determined with the help of the symbolic software MATLAB [36]. The expressions of the displacements including vertical deflection and bending rotation are shown in Appendix A.

According to Eq. (11), the obtained parametric solution of motion equations (Eq. (1)) in the local coordinate contains four unknown coefficients (a_0, a_1) and (b_0, b_1). It has to be notified that if the mentioned undefined terms are considered functions of the displacements of degree of freedom (DOF), then all the remaining coefficients $a_i, b_i\ (i=2,3,4,..)$ also become functions of the displacements of DOF. The expressions of the angle of rotation of the cross-section and vertical displacement ($\theta(\varepsilon)$ and $w(\varepsilon)$) can thus be derived as a function of the displacement of DOF. Finally, the natural frequencies of transverse vibration are computed by imposing the natural boundary conditions corresponding to a single span Timoshenko beam (two at each end of the beam) and solving the eigenvalue problem.

In the present method, hinged-hinged, clamped-free and clamped-clamped beams are surveyed. Their corresponding boundary conditions at the left end (*x=0*) and the right one (*x=L*) of the beam can be written as:

- ***For simply supported members**:*

1- At $x=0\ (\varepsilon=0)$,

Thus:

$$w(0)=0\rightarrow b_0w_0(0)+b_1w_1(0)+b_0w_2(0)+b_1w_3(0)=0$$
$$+b_0w_2(0)+b_1w_3(0)=0 \quad (12a)$$

and

$$\left.\frac{d\theta(x)}{dx}\right|_{x=0}=\frac{1}{L}\left.\frac{d\theta(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0}=0$$
$$\rightarrow a_0\frac{\theta_0'(0)}{L}+a_1\frac{\theta_1'(0)}{L}+b_0\frac{\theta_2'(0)}{L}+b_1\frac{\theta_3'(0)}{L}=0 \quad (12b)$$

2- At $x=L\ (\varepsilon=1)$,

As a result:

$$w(L)=w(1)=0 \rightarrow b_0 w_0(1)+b_1 w_1(1) + b_0 w_2(1)+b_1 w_3(1)=0 \qquad (13a)$$

and

$$\left.\frac{d\theta(x)}{dx}\right|_{x=L}=\frac{1}{L}\left.\frac{d\theta(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=1}=0 \rightarrow a_0\frac{\theta_0'(1)}{L}+a_1\frac{\theta_1'(1)}{L}+b_0\frac{\theta_2'(1)}{L}+b_1\frac{\theta_3'(1)}{L}=0 \qquad (13b)$$

- ***For cantilever members:***

1- At $x=0$ ($\varepsilon=0$), Fixed end condition in dimensionless coordinate leads to:

$$w(0)=0 \rightarrow b_0 w_0(0)+b_1 w_1(0)+b_0 w_2(0)+b_1 w_3(0)=0 \qquad (14a)$$

and

$$\theta(0)=0 \rightarrow b_0\theta_0(0)+b_1\theta_1(0)+b_2\theta_2(0)+b_3\theta_3(0)=0 \qquad (14b)$$

2- At $x=L$ ($\varepsilon=1$), Free end condition in dimensionless coordinate results in:

$$\left.\frac{(kGA-P)}{kGA}\frac{dw(x)}{dx}-\theta(x)\right|_{x=L} = \left.\frac{1}{L}\frac{(kGA-P)}{kGA}\frac{dw(\varepsilon)}{d\varepsilon}-\theta(\varepsilon)\right|_{\varepsilon=1}=0$$

$$\rightarrow a_0\left(\frac{kGA-P}{kGA}\frac{w_0'(1)}{L}-\theta_0(1)\right) + a_1\left(\frac{kGA-P}{kGA}\frac{w_1'(1)}{L}-\theta_1(1)\right) + b_0\left(\frac{kGA-P}{kGA}\frac{w_2'(1)}{L}-\theta_2(1)\right) + b_1\left(\frac{kGA-P}{kGA}\frac{w_3'(1)}{L}-\theta_3(1)\right)=0 \qquad (15a)$$

and

$$\left.\frac{d\theta(x)}{dx}\right|_{x=L}=\frac{1}{L}\left.\frac{d\theta(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=1}=0 \rightarrow a_0\frac{\theta_0'(1)}{L}+a_1\frac{\theta_1'(1)}{L}+b_0\frac{\theta_2'(1)}{L}+b_1\frac{\theta_3'(1)}{L}=0 \qquad (15b)$$

- ***For clamped-clamped members:***

1- At $x=0$ ($\varepsilon=0$),

$$w(0)=0 \rightarrow b_0 w_0(0)+b_1 w_1(0)+b_0 w_2(0)+b_1 w_3(0)=0 \qquad (16a)$$

and

$$\theta(0)=0 \rightarrow b_0\theta_0(0)+b_1\theta_1(0)+b_2\theta_2(0)+b_3\theta_3(0)=0 \qquad (16b)$$

2- At $x=L$ ($\varepsilon=1$),

$$w(1)=0 \rightarrow b_0 w_0(1)+b_1 w_1(1)+b_0 w_2(1)+b_1 w_3(1)=0 \qquad (17a)$$

and

$$\theta(1)=0 \rightarrow b_0\theta_0(1)+b_1\theta_1(1)+b_2\theta_2(1)+b_3\theta_3(1)=0 \qquad (17b)$$

Regarding the author's knowledge on Power Series Method (PSM) (Asgarian *et al.*, 2013 [2]; Soltani *et al.*, 2014 [29]; Soltani and Mohri, 2016 [30]), the results of this numerical approach are extremely sensitive to the number of terms considered in the power series approximations. Therefore, in each case, it is important to estimate appropriate number of terms (n) in power series expansion in order to obtain an explicit expression for deformation shape of the member and then to calculate the lowest circle frequencies with the desired precision, as well as optimizing the symbolic mathematical procedures. One can check that the vibration modes and their relating frequencies (ω) are highly dependent on the number of terms in power series expansion. The values of free vibration frequencies of higher modes approach the exact solutions as the number of power series expansion increases. This can be observed in [29] (Soltani *et al.*, 2014). It is noteworthy that the values obtained for the natural frequency (ω) are real values.

3. Numerical Examples

In this section, to illustrate the accuracy, efficiency and reliability of the proposed numerical method, based on power series expansions, two comprehensive examples relating to non-homogeneous Timoshenko beams with non-uniform cross-section are included and the results are compared with those obtained by finite element simulations using ABAQUS software (Hibbitt *et al.*, 2011 [12]) and other available benchmark solutions.

In the following, the mechanical properties at the left support ($x=0$, $\varepsilon=0$) and the right one ($x=L$, $\varepsilon=1$) of the beam are respectively indicated with the subscripts 0 and 1, relating to the dimensionless coordinate system ($\varepsilon=x/L$). In order to simplify the solution procedure and the illustration of obtained results, two non-dimensional parameters are also adopted as:

$$r=\frac{I_0}{A_0L^2} \qquad (18a)$$

$$\omega_{nor}=\omega\times\sqrt{\frac{\rho_0A_0L^4}{E_0I_0}} \qquad (18b)$$

Where $I_0=\frac{h_0b_0^3}{12}$ and $A_0=b_0h_0$.

In the following numerical examples, the natural frequencies for three sets of non-prismatic Timoshenko members with different boundary conditions (fixed-fixed, fixed–free and hinged–hinged) are calculated. The cross-section of all considered beams is in the form of a rectangle.

In the first case (**Case A**), the height of the beam's section is (b_0) at the left support and linearly decreased to ($b_1=(1-\beta)b_0$) at the other end. The width of the beam (h_0) remains constant. Therefore, the acquired section at the left end becomes a rectangle with a height of (b_1) and width of (h_0).

In **Case B**, the height and breadth of the beam are concurrently allowed to vary linearly along the member's length with same tapering ratio (β) to $b_1=(1-\beta)b_0$ and $h_1=(1-\beta)h_0$, respectively.

In the third case (**Case C**), the height and width of beam's section at the left support are respectively made to diminish linearly to $b_1=(1-\beta)b_0$ and $h_1=(1-\alpha)h_0$ at the other end with different tapering ratios (α and β).

In this study, the tapering parameters (α and β) can change from zero (prismatic beam) to a range of [0.1, 0.9] for non-uniform beams. For the above-mentioned cases, the minor axis moment of inertia and the cross-section area in the local coordinate can be respectively represented in the following forms:

Case A: $I(\varepsilon)=I_0(1-\beta\varepsilon)^3;A(\xi)=A_0(1-\beta\varepsilon)$ (19a)

Case B: $I(\varepsilon)=I_0(1-\beta\varepsilon)^4;A(\varepsilon)=A_0(1-\beta\varepsilon)^2$ (19b)

Case C: $I(\varepsilon)=I_0(1-\alpha\varepsilon)(1-\beta\varepsilon)^3;A(\varepsilon)=A_0(1-\alpha\varepsilon)(1-\beta\varepsilon)$ (19c)

It should be noted that Poisson's ratio and the shear correction factor in all presented cases are assumed to be 0.3 and 5/6, respectively.

3.1. Example 1- Non-prismatic Timoshenko beam with polynomial distributions of material properties

To study the effects of material non-homogeneity on linear transverse vibration behavior, it is supposed that the beam is made of two different materials specifically aluminum (***Al***) and zirconia (***ZrO₂***) in the length direction with the following characteristics:

ZrO_2: E_0=200GPa, ρ_0=5700 kg/m^3;

Al: E_1=70GPa, ρ_1=2702 kg/m^3;

The variation of Young's modulus of elasticity and the material density along the beam axis are defined with the following power-law formulations:

$$E(\varepsilon)=E_0+(E_1-E_0)\varepsilon^m \quad (20a)$$

$$\rho(\varepsilon)=\rho_0+(\rho_1-\rho_0)\varepsilon^m \quad (20b)$$

In Eq. (20), ***m*** signifies the material non-homogeneity index. By notifying this formulation, it can be concluded that by descending the gradient index (***m***), the proportion of zirconia over the beam's length decreases. For convenience, ***m*** is assumed to be a positive integer. Therefore, PSM can be easily applied to the problem. In this example, the geometry of the member is contemplated according to the first two cases: **A** and **B** and by setting r equals to *0.01* (Eq. (18a)).

The aim of the first section of this example is to define the required number of terms in power series expansions to obtain an acceptable accuracy on natural frequencies. Regarding this, the first two non-dimensional transverse frequencies (ω_{nor}) for two cases (**A** and **B**), various values of tapering ratios (β = 0.2 and 0.5) and different boundary conditions are calculated with respect to the number of terms (n) considered in the power series expansions. The convergence study is carried out for both homogeneous and non-homogeneous Timoshenko beams by contemplating the gradient parameter (***m***) equals to two. The obtained results by the proposed numerical technique have been compared with those acquired by finite element method using ABAQUS software and with exact ones obtained using FEM developed by Shahba *et al.*, 2011 [25] in Table 1.

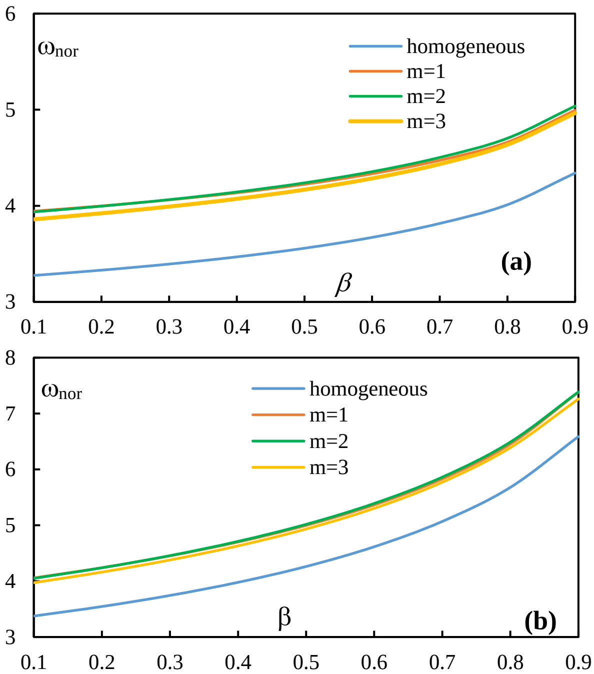

Fig. 2: Variations of the dimensionless fundamental natural frequency of cantilever AFG non-prismatic Timoshenko beam with respect to taper ratio: (a) Case A: cubic variation of moment of inertia, (b) Case B: quartic variation of moment of inertia

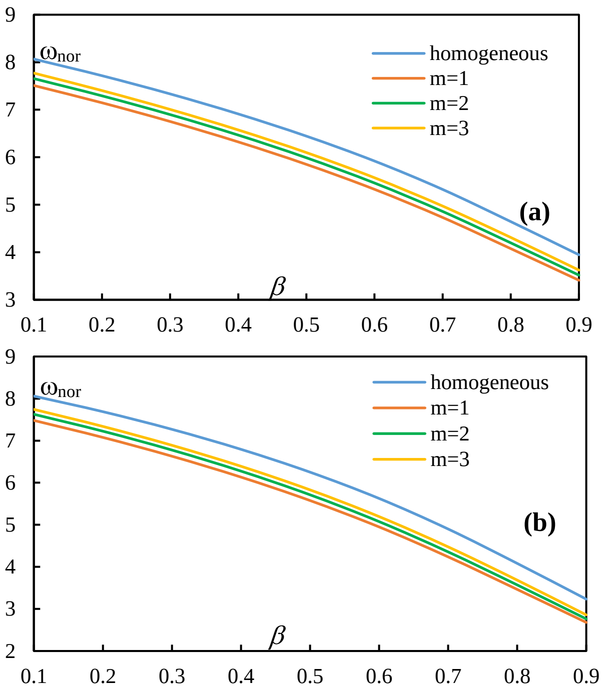

Fig. 3: Variations of the dimensionless fundamental natural frequency of simply supported AFG non-prismatic Timoshenko beam with respect to taper ratio: (a) Case A: cubic variation of moment of inertia, (b) Case B: quartic variation of moment of inertia

Table 1: Convergence of the proposed numerical technique in determination of the first two non-dimensional transverse frequencies (ω_{nor}) for axially homogeneous and non-homogeneous Timoshenko beam with two different end conditions

Case	Boundary Conditions	Taper ratio (β)	Material	Mode	Number of terms of power series (n)						Reference
					10	20	30	40	50	60	
A	**Fixed-Free**	0.2	Homogenous	1	3.3241	3.3307	3.3307	3.3307	3.3307	3.3307	3.3770[12]
				2	13.7643	14.2892	14.2892	14.2892	14.2892	14.2892	14.6120[12]
			m=2	1	4.2462	4.0084	3.9963	3.9955	3.9955	3.9954	3.9956[25]
				2	15.2479	15.0981	15.0315	15.0216	15.0203	15.0201	15.0247[25]
		0.5	Homogenous	1	3.3681	3.5583	3.5591	3.5591	3.5591	3.5591	3.6890[12]
				2	26.4336	13.7683	13.7753	13.7754	13.7754	13.7754	14.1280[12]
			m=2	1	4.2720	4.2425	4.2390	4.2386	4.2386	4.2385	4.2393[25]
				2	14.1372	14.4309	14.4247	14.4223	14.4219	14.4219	14.4282[25]
	Hinged-Hinged	0.2	Homogenous	1	7.7001	7.7160	7.7160	7.7160	7.7160	7.7160	7.7370[12]
				2	21.1779	24.0858	24.0868	24.0868	24.0868	24.0868	24.1340[12]
			m=2	1	7.4702	7.3094	7.2923	7.2904	7.2901	7.2901	7.2921[25]
				2	21.8149	23.1395	23.1146	23.1117	23.1113	23.1113	24.1380[25]
		0.5	Homogenous	1	6.4839	6.4445	6.4442	6.4442	6.4442	6.4442	6.4740[12]
				2	27.0793	21.5906	21.5893	21.5893	21.5893	21.5893	21.6730[12]
			m=2	1	6.3442	6.0236	5.9891	5.9850	5.9845	5.9845	5.9897[25]
				2	22.6616	20.8746	20.8202	20.8134	20.8126	20.8125	20.8350[25]
B	**Fixed-Free**	0.2	Homogenous	1	3.5381	3.5462	3.5462	3.5462	3.5462	3.5462	3.6044[12]
				2	14.7427	14.6074	14.6074	14.6074	14.6074	14.6074	14.9245[12]
			m=2	1	4.4366	4.2498	4.2390	4.2382	4.2381	4.2381	4.2384[25]
				2	15.8573	15.4062	15.3488	15.3404	15.3393	15.3392	15.3441[25]
		0.5	Homogenous	1	4.0502	4.2609	4.2628	4.2628	4.2628	4.2628	4.3700[12]
				2	13.6635	14.6610	14.6736	14.6736	14.6736	14.6736	14.9510[12]
			m=2	1	5.0492	5.0201	5.0170	5.0164	5.0164	5.0164	5.0178[25]
				2	14.9397	15.3362	15.3412	15.3403	15.3401	15.3401	15.3573[25]
	Hinged-Hinged	0.2	Homogenous	1	7.6787	7.6887	7.6887	7.6887	7.6887	7.6887	7.7090[12]
				2	22.0085	24.0908	24.0913	24.0913	24.0913	24.0913	24.1380[12]
			m=2	1	7.4139	7.2427	7.2246	7.2225	7.2223	7.2222	7.2245[25]
				2	22.2912	23.1456	23.1192	23.1160	23.1157	23.1156	23.1398[25]
		0.5	Homogenous	1	6.3512	6.2518	6.2508	6.2508	6.2508	6.2508	6.2760[12]
				2	31.1362	21.6495	21.6462	21.6462	21.6462	21.6462	21.7210[12]
			m=2	1	6.1597	5.7570	5.7128	5.7076	5.7070	5.7069	5.7118[25]
				2	22.4540	20.9708	20.9022	20.8936	20.8925	20.8924	20.9191[25]

Table 2: First two normalized natural frequencies (ω_{nor}) of homogeneous non-prismatic Timoshenko beam for various tapering ratios and three different end conditions

End Conditions	β	Case A				Case B			
		PSM		ABAQUS		PSM		ABAQUS	
		ω_{nor_1}	ω_{nor_2}	ω_{nor_1}	ω_{nor_2}	ω_{nor_1}	ω_{nor_2}	ω_{nor_1}	ω_{nor_2}
Fixed-Free	**0.1**	3.2756	14.3931	3.4270	14.7040	3.3758	14.5485	3.5280	14.8630
	0.2	3.3307	14.2892	3.3770	14.6120	3.5462	14.6074	3.6044	14.9245
	0.3	3.3941	14.1540	3.4331	14.5706	3.7444	14.6459	3.8890	14.9890
	0.4	3.4689	13.9841	3.6050	14.3280	3.9789	14.6661	4.1760	15.1710
	0.5	3.5591	13.7753	3.6890	14.1280	4.2628	14.6736	4.3700	14.9510
	0.6	3.6716	13.5235	3.7970	13.8850	4.6159	14.6799	4.7220	14.9620
	0.7	3.8170	13.2163	3.9400	13.5970	5.0690	14.6988	5.1770	14.9990
	0.8	4.0115	12.8090	4.1420	13.2750	5.6787	14.7883	5.7960	15.1470
	0.9	4.3414	12.5482	4.4580	12.9770	6.5879	15.5197	6.6870	15.6730
Hinged-Hinged	**0.1**	8.0662	24.7480	8.0580	24.7890	8.0596	24.7489	8.0790	24.7890
	0.2	7.7160	24.0868	7.7370	24.1340	7.6887	24.0913	7.7090	24.1380
	0.3	7.3328	23.3509	7.3560	23.4080	7.2691	23.3638	7.2910	23.4180
	0.4	6.9114	22.5253	6.9380	22.5930	6.7933	22.5541	6.8360	22.6690
	0.5	6.4442	21.5893	6.4740	21.6730	6.2508	21.6462	6.2760	21.7210
	0.6	5.9199	20.5130	5.9550	20.6180	5.6260	20.6185	5.6510	20.7050
	0.7	5.3221	19.2531	5.3900	19.4870	4.8997	19.4504	4.9120	19.5340
	0.8	4.6449	17.7894	4.6530	17.8940	4.0822	18.2193	4.0010	18.1420
	0.9	3.9471	16.3529	3.7270	15.9500	3.2331	17.2697	2.7230	16.3300
	0.1	13.5478	28.1785	13.5610	28.1940	13.5499	28.1804	13.5630	28.1960

Fixed-Fixed	**0.2**	13.2223	27.7782	13.2380	27.7980	13.2316	27.7869	13.2470	27.8060
	0.3	12.8504	27.3007	12.8680	27.3260	12.8739	27.3230	12.8910	27.8001
	0.4	12.4222	26.7233	12.4440	26.7550	12.4692	26.7693	12.4890	26.7990
	0.5	11.9235	26.0140	11.9500	26.0560	12.0077	26.0987	12.0450	26.1580
	0.6	11.3342	25.1249	11.3680	25.1820	11.4759	25.2723	11.5040	25.3210
	0.7	10.6238	23.9827	10.6680	24.0620	10.8590	24.2373	10.8880	24.2940
	0.8	9.7640	22.4956	9.9230	22.8030	10.1857	22.9803	10.1540	22.9560
	0.9	8.8493	20.7515	8.6600	20.4500	9.6445	21.7649	9.2430	21.1020

Table 3: First four non-dimensional transverse frequencies of an axially FG tapered Timoshenko beam (*Case* **A**; ***m=2***)

End Conditions	β	**PSM**				**Shahba *et al.*, 2011 [25]**			
		ω_{nor_1}	ω_{nor_2}	ω_{nor_3}	ω_{nor_4}	ω_{nor_1}	ω_{nor_2}	ω_{nor_3}	ω_{nor_4}
Fixed-Free	**0.1**	3.9367	15.1691	31.2133	47.5831	3.9359	15.1577	31.2638	47.7164
	0.2	3.9963	15.0315	30.7644	47.2999	3.9956	15.0247	30.8092	47.4362
	0.3	4.0645	14.8647	30.2438	46.8099	4.064	14.8611	30.286	46.9481
	0.4	4.1441	14.6642	29.6395	46.1213	4.1438	14.6636	29.6818	462610
	0.5	4.2390	14.4247	28.9325	45.2184	4.2393	14.4282	28.9791	45.3634
	0.6	4.3557	14.1390	28.0895	44.0516	4.3571	14.1513	28.1532	44.2201
	0.7	4.5037	13.7902	27.0347	42.4940	4.5090	13.8314	27.1684	42.763
	0.8	4.7033	13.3594	25.5978	40.1103	4.7180	13.4793	25.9735	40.8666
	0.9	5.0387	13.1333	24.2837	37.5732	5.0371	13.1700	24.5212	38.2968
Hinged-Hinged	**0.1**	7.6547	23.7153	41.7046	57.7388	7.6545	23.7369	41.821	57.8739
	0.2	7.2923	23.1146	41.0094	59.1739	7.2921	23.1346	41.1203	59.4848
	0.3	6.8978	22.4418	40.2002	58.4650	6.8965	22.4604	40.3053	58.7642
	0.4	6.4659	21.6832	39.2519	57.4616	6.4653	21.7002	39.3507	57.7558
	0.5	5.9891	20.8202	38.1274	56.2303	5.9879	20.8350	38.2191	56.5105
	0.6	5.4572	19.8273	36.7724	54.6960	5.454	19.8372	36.8536	54.9547
	0.7	4.8588	18.6746	35.1070	52.7350	4.8448	18.6634	35.1618	52.9529
	0.8	4.1974	17.3737	33.0537	50.1271	4.1244	17.2364	32.975	50.2463
	0.9	3.5183	16.1334	30.9399	47.1922	3.2016	15.3775	29.9011	46.2153
Fixed-Fixed	**0.1**	12.4651	26.3822	42.9600	59.3072	12.4689	26.4153	43.0904	59.6829
	0.2	12.2086	26.0701	42.4941	59.1799	12.2126	26.1023	42.6211	59.4873
	0.3	11.9130	25.6923	41.9527	58.7677	11.9172	25.7236	42.0759	59.0576
	0.4	11.5697	25.2288	41.3095	58.1325	11.5739	25.2590	41.4283	58.4306
	0.5	11.1664	24.6511	40.5240	57.2971	11.1706	24.6797	40.6374	57.5925
	0.6	10.6865	23.9170	39.5325	56.1983	10.6896	23.9424	39.6375	56.4814
	0.7	10.1080	22.9657	38.2334	54.7086	10.1036	22.9773	38.3161	54.9627
	0.8	9.4236	21.7335	36.4770	52.6149	9.3634	21.6575	36.4588	52.7499
	0.9	8.7203	20.3117	34.2502	49.6266	8.3590	19.6945	33.5615	49.1234

After noticing the results tabulated in Table 1, it can be stated that for a satisfactory result, especially at higher modes, the minimum number of terms for the power series (***n***) must be 30. In the case of taper ratios higher than 0.8, it is worthy to note that 50 terms of power series must be taken into account to calculate the values of free transverse frequencies with an acceptable error rate.

In what follows, the precision of the proposed numerical approach is verified by applying to homogenous non-prismatic beams (Cases **A** & **B**). The first two natural frequencies for various types of boundary conditions and different values of taper ratios (β) are tabulated in Table 2. It can be recognized that the results are in an excellent agreement with those acquired by ABAQUS software. In the case of fixed-fixed and hinged-hinged, it is also obvious that the natural frequency decreases with an increase in taper ratio, which has resulted from the reduction in cross-sectional area and moment of inertia and consequently mass and stiffness of the elastic member. In order to evaluate the values of dimensionless natural frequency parameters with an acceptable error rate, 30 terms of power series have been taken into account.

The dimensionless natural frequencies (ω_{nor}) relating to the first four vibrational modes of AFG non-prismatic beam for a non-homogeneity index of ***m=2*** have been arranged in Tables 3-4. Table 3 consists of the results of case A, while Table 4 summarizes the results of case B. The obtained results by the proposed numerical technique have been compared with those reported in [25].

After reviewing the results presented in Tables 2 to 4, it can be stated that the current procedure based on power series expansions is applicable for the higher vibration modes, as well as for the first one with excellent accuracy. Subsequently, the influences of tapering parameter (β) on dimensionless natural frequencies of FG non-prismatic beam by considering the non-homogeneity index **m**=1, 2 and 3 for fixed-free, simply supported and fixed-fixed are respectively illustrated in Figs. 2 to 4.

Table 4: First four non-dimensional transverse frequencies of an axially FG Timoshenko beam (*Case* **B**; ***m=2***)

End Conditions	β	PSM				Shahba *et al.*, 2011 [25]			
		ω_{nor_1}	ω_{nor_2}	ω_{nor_3}	ω_{nor_4}	ω_{nor_1}	ω_{nor_2}	ω_{nor_3}	ω_{nor_4}
Fixed-Free	**0.1**	4.0500	15.3227	31.3305	47.6892	4.0492	15.3129	31.3795	47.8232
	0.2	4.2390	15.3488	31.0109	47.5171	4.2384	15.3441	31.0546	47.655
	0.3	4.4569	15.3591	30.6379	47.1495	4.4565	15.3579	30.6798	47.2907
	0.4	4.7121	15.3550	30.2060	46.6035	4.7121	15.3573	30.2496	46.7483
	0.5	5.0170	15.3412	29.7063	45.8744	5.0178	15.3487	29.7571	46.0287
	0.6	5.3904	15.3272	29.1206	44.9278	5.3931	15.3467	29.1962	45.1172
	0.7	5.8617	15.3291	28.3980	43.6495	5.8695	15.3856	28.568	43.9835
	0.8	6.4890	15.4500	27.5444	41.7703	6.5009	15.5568	27.9102	42.5941
	0.9	7.3877	16.2212	27.5672	40.9326	7.3787	16.1537	27.4806	41.0442
Hinged-Hinged	**0.1**	7.6269	23.7152	41.7024	57.7920	7.6269	23.7369	41.8193	57.9246
	0.2	7.2246	23.1192	41.0122	59.2204	7.2245	23.1398	41.1243	59.5225
	0.3	6.7765	22.4586	40.2175	58.4758	6.7764	22.4783	40.3248	58.7787
	0.4	6.2757	21.7236	39.2970	57.4878	6.2755	21.7423	39.3994	57.7881
	0.5	5.7128	20.9022	38.2207	56.2962	5.7118	20.9191	38.3174	56.5841
	0.6	5.0758	19.9820	36.9481	54.8345	5.0709	19.9917	37.0337	55.1026
	0.7	4.3556	18.9660	35.4340	53.0095	4.3295	18.9367	35.4785	53.2304
	0.8	3.5706	17.9552	33.7316	50.7399	3.4452	17.7204	33.5322	50.7616
	0.9	2.7630	17.2371	32.3439	48.6833	2.3193	16.2962	30.9578	47.2473
Fixed-Fixed	**0.1**	12.4772	26.4043	42.9790	59.3417	12.4812	26.4376	43.1098	59.7067
	0.2	12.2385	26.1194	42.5353	59.2401	12.2429	26.1524	42.6634	59.5335
	0.3	11.9689	25.7762	42.0208	58.8343	11.9737	25.8089	42.1461	59.1273
	0.4	11.6631	25.3581	41.4118	58.2223	11.6683	25.3905	41.5341	58.5279
	0.5	11.3145	24.8425	40.6731	57.4215	11.3199	24.8744	40.7919	57.7257
	0.6	10.9163	24.1982	39.7513	56.3718	10.92	24.227	39.8636	56.6668
	0.7	10.4689	23.3896	38.5698	54.9693	10.4579	23.3976	38.657	55.2354
	0.8	10.0208	22.4278	37.0543	53.0603	9.9207	22.301	37.005	53.1931
	0.9	9.7257	21.5393	35.4151	50.5748	9.301	20.7781	34.554	49.9856

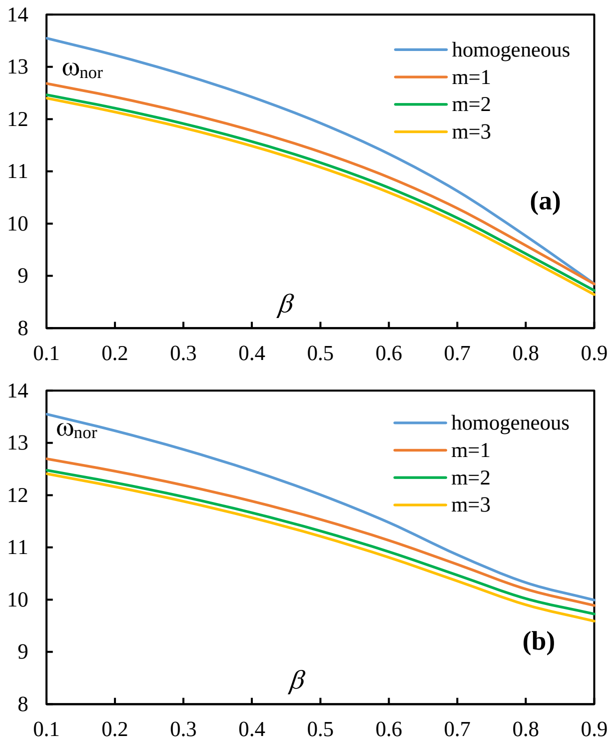

Fig. 4: Variations of the dimensionless fundamental natural frequency of fixed-fixed AFG non-prismatic Timoshenko beam with respect to taper ratio: (a) Case A: cubic variation of moment of inertia, (b) Case B: quartic variation of moment of inertia

Regarding Eq. (20), it is obvious that by increasing the gradient index (***m***) from 1 to 3, the volume fraction of Zirconia is increased and as a result, a stiffer and also heavier beam is acquired. Besides, as taper ratio is increased from 0.1 to 0.9, the moment of inertia and cross-sectional area are reduced and accordingly both stiffness and mass of beam are gradually diminished. Therefore, the beam easily becomes weaker and lighter. Since the free transverse vibration in the absence of damping is dependent on both the mentioned terms (mass and stiffness), it is not possible to predict the variations of natural frequency with respect to non-homogeneity index (***m***) and tapering parameter (***β***) concurrently. This fact can be easily observed in Figs. 2-4.

3.2. Example 2- Axially functionally graded tapered Timoshenko beam under axial load

Natural frequencies and vibrational mode shapes of structural members subjected to a compressive or tensile concentrated axial load are key factors in the design of different structures ranging from civil engineering to mechanical constituents. Therefore, exact calculation of bending vibration characteristics of flexural members could considerably help design engineers to improve the quality and the performance of structures. To study the effect of axial load on free vibration behavior of Timoshenko beam, another example is presented. In this regard, an elastic beam with rectangle cross-section whose height tapers linearly along the member axis is taken into account. Therefore, the cross-sectional area and moment of inertia are identical to Case ***A*** of the first example.

Table 5: First non-dimensional vibration frequency (ω_{nor}) for Timoshenko beam subjected to a concentrated axial load at its ends

Boundary Conditions	Taper ratio (β)	Material	Compressive axial load			Tensile axial load		
			+0.2	**+0.5**	**+0.8**	**-0.2**	**-0.5**	**-0.8**
Fixed-Free	**0**	Homogenous	3.1457	2.6045	1.8544	3.7094	4.0605	4.3699
		m=2	3.7542	3.1076	2.1540	4.4016	4.7945	5.1359
	0.1	Homogenous	3.1240	2.5695	1.7841	3.7516	4.1026	4.4114
		m=2	3.7992	3.1489	2.1788	4.4452	4.8353	5.1733
	0.3	Homogenous	3.2884	2.7339	1.9488	3.8565	4.2059	4.5116
		m=2	3.9142	3.2612	2.2589	4.5505	4.9301	5.2560
	0.5	Homogenous	3.4433	2.8879	2.0874	4.0034	4.3442	4.6404
		m=2	4.0853	3.4448	2.4222	4.6932	5.0501	5.3549
	0.7	Homogenous	3.7128	3.1899	2.4232	4.2312	4.5430	4.8124
		m=2	4.3774	3.8005	2.8264	4.9066	5.2121	5.4714
Hinged-Hinged	**0**	Homogenous	7.5362	5.9698	3.8042	9.2186	10.3010	11.2794
		m=2	7.1964	5.7341	3.7004	8.7594	9.7620	10.6669
	0.1	Homogenous	7.2550	5.7575	3.6930	8.8663	9.9032	10.8408
		m=2	6.9012	5.5056	3.5600	8.3901	9.344	10.2043
	0.3	Homogenous	6.6120	5.2591	3.3947	8.0655	9.0003	9.8452
		m=2	6.2339	4.9913	3.2492	7.5528	8.3946	9.1521
	0.5	Homogenous	5.8396	4.6698	3.0566	7.0923	7.8959	8.6208
		m=2	5.4311	4.3725	2.8759	6.5458	7.2528	7.8864
	0.7	Homogenous	4.8697	3.9404	2.6586	5.8587	6.4891	7.0441
		m=2	4.4146	3.5852	2.3977	5.2762	5.8167	6.2975

The natural frequencies are also carried out for two cases: axially non-homogeneous and homogeneous beams. In the case of axially FG member, the distribution of modulus of elasticity and density of material are contemplated to vary in the longitudinal direction with a power-law formulation as expressed in Eq. (20). In this case, the material non-homogeneity parameter (***m***) is assumed to be equal to 2. Moreover, the vibration frequency is acquired for five different tapering parameters: β=0, 0.1, 0.3, 0.5 and 0.7. The following non-dimensional quantity corresponding to point external axial force is also used:

$$\lambda = \frac{P}{P_{cr}} \qquad (21)$$

Non-dimensional critical loads (P_{cr}) of the contemplated members made of homogenous material are obtained using ABAQUS software and those related to axially functionally graded beams are stated in Shaba et al., 2011 [25]. In the case of compressive axial force, λ is negative, however; this parameter is positive for a tensile load. The vibration frequencies are obtained for $\lambda = \pm 0.2, \pm 0.5, \pm 0.8$. It is noteworthy that non-dimensional free vibration frequencies are computed for homogenous and AFG beams with simply supported and clamped-free boundary conditions. Results are tabulated in Table 5.

In case of tensile axial load, fundamental frequency is observed to increase with an increase in λ. This is due to the fact that the free transverse vibration in the absence of damping and harmonic external loads is directly proportional to stiffness. In general, the effect of a tensile axial load is to decrease the deflection and increase stiffness of the member and consequently the natural frequencies, as can be seen from the results presented in Table 5. While in case of compressive axial force beam, natural frequency is observed to decrease with an increase in normalized axial force parameter (λ). As compressive axial force approaches the critical buckling load, the transverse deflection of the member increases noticeably and as a result, the stiffness of the member vanishes and a weaker member is obtained. Therefore, a significant decrease in vibration frequency of the member is observed (see Table 5).

3.3. Example 3- Non-prismatic Timoshenko beam with exponential distribution of material properties

In this example, free vibration analysis is accomplished for an AFG double-tapered Timoshenko beam whose cross-sectional area and moment of inertia vary over the member's length according to Case **C** (Eq. (19c)). For such a member, it is also assumed that the material properties including Young's modulus of elasticity and the mass density change exponentially along the beam as:

$$E(\varepsilon) = E_0 e^{-0.5\varepsilon} \qquad (22a)$$

$$\rho(\varepsilon) = \rho_0 e^{-0.5\varepsilon} \qquad (22b)$$

FGM possesses properties that vary exponentially through longitudinal direction is one of the most special cases of composite material that few numerical methods are able to solve its governing differential equation. Based on the power series method, all variable terms in motion equations are presented in a polynomial form. In the presence of exponential variations, the Maclaurin series expansion should be adopted. The explicit forms of Maclaurin series for material properties (Eq. (22)) are thus defined as:

$$E(\varepsilon) = E_0 \sum_{i=0}^{\infty} \frac{(-0.5)^i}{i\,!} \varepsilon^i \qquad (23a)$$

$$\rho(\varepsilon) = \rho_0 \sum_{i=0}^{\infty} \frac{(-0.5)^i}{i\,!} \varepsilon^i \qquad (23b)$$

The first two non-dimensional free frequency parameters for different combinations of height and width tapering ratios (α and β) with two types of end conditions are tabulated in Table 6.

Table 6: First two dimensionless natural frequencies (ω_{nor}) for exponential FGM double tapered Timoshenko beam with two different end conditions

End conditions	Height taper ratio (β)	Width taper ratio (α)									
		0.0		0.2		0.4		0.6		0.8	
		ω_{nor_1}	ω_{nor_2}	ω_{nor_1}	ω_{nor_2}	ω_{nor_1}	ω_{nor_2}	ω_{nor_1}	ω_{nor_2}	ω_{nor_1}	ω_{nor_2}
Hinged-Hinged	0.0	7.987	24.252	7.930	24.242	7.831	24.213	7.652	24.133	7.273	23.861
	0.2	7.290	23.111	7.222	23.116	7.115	23.110	6.934	23.073	6.572	22.898
	0.4	6.463	21.678	6.386	21.698	6.272	21.717	6.089	21.726	5.746	21.653
	0.6	5.450	19.814	5.366	19.853	5.247	19.902	5.064	19.962	4.741	20.003
	0.8	4.119	17.211	4.033	17.277	3.914	17.367	3.737	17.497	3.441	17.692
Cantilever	0.0	3.883	15.259	4.124	15.596	4.444	16.027	4.898	16.629	5.614	17.638
	0.2	3.995	15.020	4.238	15.339	4.560	15.750	5.017	16.329	5.740	17.316
	0.4	4.143	14.658	4.387	14.960	4.711	15.351	5.171	15.908	5.902	16.875
	0.6	4.356	14.143	4.601	14.428	4.927	14.799	5.391	15.335	6.128	16.284
	0.8	4.715	13.457	4.962	13.723	5.291	14.075	5.756	14.589	6.494	15.520

Table 7: The first three normalized natural frequencies of clamped-free Timoshenko beam and Euler-Bernoulli beam

β	Timoshenko Beam Theory				Euler-Bernoulli Beam Theory		
	L/b_0	ω_{nor_1}	ω_{nor_2}	ω_{nor_3}	ω_{nor_1}	ω_{nor_2}	ω_{nor_3}
0.2	**5**	3.7420	18.0348	41.9204	**3.8551** 3.8551	**21.0571** 21.0568	**56.6379[3]** 56.6303[6]
	10	3.8257	20.1527	51.3881			
	50	3.8539	21.0180	56.3842			
	100	3.8548	21.0470	56.5684			
0.4	**5**	4.1948	17.622	39.7719	**4.3187** 4.3187	**20.0505** 20.0500	**51.3423[3]** 51.3346[6]
	10	4.2866	19.3428	47.4121			
	50	4.3174	20.0200	51.1542			
	100	4.3184	20.0424	51.2895			
0.5	**5**	4.4931	17.3940	38.5312	**4.6251** 4.6252	**19.5482** 19.5476	**48.5870[3]** 48.5789[6]
	10	4.5909	18.9289	45.2611			
	50	4.6237	19.5214	48.4274			
	100	4.6247	19.5407	48.5317			

It is concluded again that the geometrical parameter is an important parameter in determining the lower-order normalized natural frequencies for simply supported and cantilever exponential FG Timoshenko beams.

3.4. Example 4- Tapered Timoshenko beam versus tapered Euler-Bernoulli beam

To depict the difference between the free vibration frequencies of Euler-Bernoulli beam theory and those of Timoshenko beam model, this illustrative example is also considered. In the case of double tapered beam, both the height and width of the beam cross-section decrease linearly along the member's axis with the same tapering ratio (β), as expressed in Eq. (19b). For different values of L/b_0, the first three natural frequencies are calculated for fixed-free beams and the results are arranged in Table 7. The values of taper parameter (β) and the type of boundary conditions are selected in such a way that make comparison possible with other available references in the case of non-prismatic Euler-Bernoulli beam (Banerjee et al., 2006 [6]; Attarnejad, 2010 [3]). From Table 7, one can see that the natural frequencies calculated by employing Euler-Bernoulli theory (EBT) are overestimated. The difference between EBT and shear deformation theory (Timoshenko beam model) is small for slender beams ($L/b_0>50$). Note that for deep members ($L/b_0<10$), this difference is significant especially at higher modes (see Table 7). This is owing to the fact that the influences of shear deformation and rotatory inertia are ignorable in the Euler-Bernoulli theory. In general, the effects of both mentioned parameters are to increase the deflection and reduce the stiffness, as well as natural frequencies.

4. Conclusions

In the present study, the free transverse vibration analysis of elastic non-prismatic Timoshenko beam made of axially functionally graded material (AFGM) with arbitrary boundary conditions has been investigated using a numerical approach. The effect of point axial load which is tangential to the longitudinal axis of beam on bending vibration frequency is also taken into consideration.

In presence of arbitrary variation in geometrical and material properties, the governing motion equations of Timoshenko beam include a pair of second-order differential equations with variable coefficients. Therefore, the vibration analysis becomes more complicated. In this regard, the solution of the system of second-order differential equations is not straightforward and only numerical methods such as power series expansions are possible. In this study, the power series expansions are applied in order to facilitate the solution of the governing motion differential equations of AFG non-prismatic Timoshenko beams. The variable terms in equation of motions, namely, cross-sectional area, moment of inertia, mechanical properties including Young's modulus of elasticity and density of material, as well as the vertical deflection and angle of rotation, are expanded into power series form. The transverse natural frequencies of the considered member are finally derived by imposing the boundary conditions and solving the eigenvalue problem.

In order to demonstrate the accuracy and performance of the proposed method, four numerical examples considering the effects of tapering, material variations and various end conditions are carried out and the obtained results are compared with accessible analytical and numerical solutions. In most cases, it can be concluded that by considering 40 terms of power series, the lowest and higher-order natural frequencies of AFG tapered Timoshenko beams can be simultaneously determined with very good accuracy. From the cases analyzed above, it can be stated that the beam subjected to a compressive axial load has the lowest vibration frequencies while the beam under a tensile axial force has the largest ones. In other words, the inclusion of compressive point load effect increases the magnitude of deflection and decreases natural frequencies. It is also observed that the Euler–Bernoulli beam theory overestimates natural frequencies of tapered homogenous beam due to neglecting the shear deformation effect. Furthermore, the difference between vibration frequencies of Euler–Bernoulli beam theory and those of Timoshenko beam theory is noticeable at higher vibrational modes, besides it is significance for high values of thickness ratio. Finally, it can be stated that Euler–Bernoulli beam model is suitable for long and slender beams with constant or variable cross-sections, whereas the TBT is a good approximation for moderately deep beams.

APPENDIX A

By using symbolic MATLAB software, the following displacements functions θ_i &w_i $(i = 0,1,2,3)$ are derived. The first few terms are expressed as:

$$
\begin{aligned}
w_0(\varepsilon) &= 1+\frac{\varepsilon^2}{2(P-kA_0^*G_0^*)}\left\{L^2\rho_0^*A_0^*\omega^2\right\} \\
&+\frac{\varepsilon^3}{6(P-kA_0^*G_0^*)}\left\{L^2\omega^2(A_0^*\rho_1^*+A_1^*\rho_0^*)-\frac{2L^2\omega^2\rho_0^*A_0^*}{A_0^*G_0^*}(A_0^*G_1^*+A_1^*G_0^*)\right\} \\
&+\frac{\varepsilon^4}{12(P-kA_0^*G_0^*)}\left\{-\frac{L^2\omega^2A_0^*\rho_0^*}{2kA_0^*G_0^*}(L^2(\omega^2A_0^*\rho_0^*+k^2A_0^{*2}G_0^{*2})+6k(G_2^*A_0^*+G_0^*A_2^*+G_1^*A_1^*))\right. \\
&\qquad +L^2\omega^2(A_0^*\rho_2^*+A_2^*\rho_0^*+A_1^*\rho_1^*) \\
&\qquad \left.-\frac{3(A_0^*G_1^*+A_1^*G_0^*)}{2A_0^*G_0^*}(L^2\omega^2(A_0^*\rho_1^*+A_1^*\rho_0^*)-\frac{2L^2(A_0^*G_1^*+A_1^*G_0^*)}{A_0^*G_0^*}\omega^2A_0^*\rho_0^*)\right\}+\ldots
\end{aligned}
$$

$$
\begin{aligned}
w_1(\varepsilon) &= \varepsilon-\frac{\varepsilon^2}{2A_0^*G_0^*}\left\{A_0^*G_1^*+A_1^*G_0^*\right\} \\
&-\frac{\varepsilon^3}{6kA_0^*G_0^*}\left\{\frac{k^2L^2A_0^{*2}G_0^{*2}}{E_0^*I_0^*}+2k(A_0^*G_2^*+A_2^*G_0^*+A_1^*G_1^*)\right. \\
&\qquad \left.+L^2\omega^2\rho_0^*A_0^*-\frac{2k(A_0^*G_1^*+A_1^*G_0^*)^2}{A_0^*G_0^*}\right\} \\
&-\frac{\varepsilon^4}{12kA_0^*G_0^*}\left\{-\frac{(A_0^*G_1^*+A_1^*G_0^*)}{2A_0^*G_0^*}(6(A_0^*G_2^*+A_2^*G_0^*+A_1^*G_1^*)+L^2\omega^2A_0^*\rho_0^*)\right. \\
&\qquad -\frac{3(A_0^*G_1^*+A_1^*G_0^*)}{2A_0^*G_0^*}(L^2\omega^2A_0^*\rho_0^*+2k(A_0^*G_2^*+A_2^*G_0^*+A_1^*G_1^*) \\
&\qquad -\frac{2k(A_0^*G_1^*+A_1^*G_0^*)^2}{A_0^*G_0^*}+\frac{k^2L^2A_0^{*2}G_0^{*2}}{E_0^*I_0^*})+3k(A_0^*G_3^*+A_3^*G_0^*+A_1^*G_2^*+A_2^*G_1^*) \\
&\qquad +L^2\omega^2(A_0^*\rho_1^*+A_1^*\rho_0^*)+\frac{k^2L^2A_0^*G_0^*}{2E_0^*I_0^*}(-2(\frac{A_0^*E_1^*G_0^*}{E_0^*}+\frac{A_0^*G_0^*I_1^*}{I_0^*})+(A_0^*G_1^*+A_1^*G_0^*) \\
&\qquad \left.-\frac{A_0^*G_0^*}{A_0^*G_0^*}(A_0^*G_1^*+A_1^*G_0^*))+\frac{3}{2E_0^*I_0^*}(A_0^{*2}G_0^*G_1^*+A_0^*A_0^*G_0^{*2})\right\}+\ldots
\end{aligned}
$$

$$
\begin{aligned}
w_2(\varepsilon) &= \frac{\varepsilon^2}{2kA_0^*G_0^*}\left\{kL(A_0^*G_1^*+A_1^*G_0^*)\right\} \\
&+\frac{\varepsilon^3}{6kA_0^*G_0^*}\left\{2kL(A_0^*G_2^*+A_2^*G_0^*+A_1^*G_1^*)+\frac{kL^3A_0^*G_0^*}{E_0^*I_0^*}(kA_0^*G_0^*-\omega^2I_0^*\rho_0^*)\right. \\
&\qquad \left.-\frac{2k^2L(A_0^*G_1^*+A_1^*G_0^*)^2}{kA_0^*G_0^*}\right\} \\
&+\frac{\varepsilon^4}{12kA_0^*G_0^*}\left\{-\frac{kL(A_0^*G_1^*+A_1^*G_0^*)}{2kA_0^*G_0^*}(6k(A_0^*G_2^*+A_2^*G_0^*+A_1^*G_1^*)+L^2\omega^2\rho_0^*A_0^*)\right. \\
&\qquad +3kL(A_0^*G_3^*+A_3^*G_0^*+A_1^*G_2^*+A_2^*G_1^*) \\
&\qquad -\frac{3k(A_0^*G_1^*+A_1^*G_0^*)}{2kA_0^*G_0^*}(2kL(A_0^*G_2^*+A_2^*G_0^*+A_1^*G_1^*)-\frac{2Lk^2(A_0^*G_1^*+A_1^*G_0^*)^2}{kA_0^*G_0^*} \\
&\qquad +\frac{kL^3A_0^*G_0^*}{E_0^*I_0^*}(kA_0^*G_0^*-\omega^2\rho_0^*I_0^*)) \\
&\qquad -\frac{LkA_0^*G_0^*}{2E_0^*I_0^*}(2L^2(\frac{E_1^*}{E_0^*}+\frac{I_1^*}{I_0^*})(kA_0^*G_0^*-\omega^2I_0^*\rho_0^*)-kL^2(A_0^*G_1^*+A_1^*G_0^*) \\
&\qquad \left.+L^2\omega^2(I_0^*\rho_1^*+I_1^*\rho_0^*)+\frac{L^2k^2A_0^*G_0^*}{kA_0^*G_0^*}(A_1^*G_0^*+A0_1^*G_1^*)\right\}+\ldots.
\end{aligned}
$$

$$
\begin{aligned}
w_3(\varepsilon) &= \frac{\varepsilon^2}{2kA_0^*G_0^*}\left\{LkA_0^*G_0^*\right\} \\
&+\frac{\varepsilon^3}{6kA_0^*G_0^*}\left\{-\frac{2Lk^2}{kA_0^*G_0^*}(A_0^*A_1^*G_0^{*2}+A_0^{*2}G_0^*G_1^*)\right. \\
&\qquad \left.+2kL(A_0^*G_1^*+A_1^*G_0^*)-\frac{kLA_0^*G_0^*}{E_0^*I_0^*}(E_0^*I_1^*+E_1^*I_0^*)\right\} \\
&+\frac{\varepsilon^4}{12kA_0^*G_0^*}\left\{-\frac{3}{2}\frac{k(A_0^*G_1^*+A_1^*G_0^*)}{kA_0^*G_0^*}(2kL(A_0^*G_1^*+A_1^*G_0^*)\right. \\
&\qquad +\frac{2kL^2}{kA_0^*G_0^*}(A_0^*A_1^*G_0^{*2}+A_0^{*2}A_1^*G_1^*)-\frac{kLA_0^*G_0^*}{E_0^*I_0^*}(E_0^*I_1^*+E_1^*I_0^*)) \\
&\qquad +3kL(A_0^*G_2^*+A_2^*G_0^*+A_1^*G_1^*)+\frac{kLA_0^*G_0^*}{2E_0^*I_0^*}(2(\frac{E_1^*}{E_0^*}+\frac{I_1^*}{I_0^*})(E_0^*I_1^*+E_1^*I_0^*) \\
&\qquad -2(E_2^*I_0^*+E_0^*I_2^*+E_1^*I_1^*)+kL^2A_0^*G_0^*-\omega^2L^2I_0^*\rho_0^*-\frac{L^2k^2A_0^{*2}G_2^{*2}}{kA_0^*G_0^*}) \\
&\qquad -\frac{kL}{2kA_0^*G_0^*}(6k(A_0^*A_2^*G_0^{*2}+A_0^{*2}A_2^*G_0^{*2}+A_0^*A_1^*G_0^*G_1^*)+L^2\omega^2A_0^{*2}G_0^*\rho_0^*) \\
&\qquad \left.-\frac{3kL(E_0^*I_1^*+E_1^*I_0^*)}{E_0^*I_0^*}(A_0^*G_1^*+A_1^*G_0^*)\right\}+\ldots
\end{aligned}
$$

$$\theta_0(\varepsilon)=\frac{\varepsilon^3}{6E_0^*I_0^*}\left(\frac{L^3A_0^*G_0^*(A_0^*\rho_0^*\omega^2)}{A_0^*G_0^*}\right)$$

$$-\frac{\varepsilon^4}{12E_0^*I_0^*}\left\{-\frac{L^3(A_0^*\rho_0^*\omega^2)}{2A_0^*G_0^*}\Big(2(A_0^*G_1^*+A_1^*G_1^*)-3(A_0^*G_0^*E_1^*+A_0^*G_0^*I_1^*)\Big)\right.$$

$$\left.-\frac{LA_0^*G_0^*}{2A_0^*G_0^*}L^2\omega^2(A_0^*\rho_1^*+A_1^*\rho_0^*)-\frac{2L^2A_0^*\rho_0^*(A_0^*G_1^*+A_1^*G_0^*)}{A_0^*G_0^*}\right\}+\ldots$$

$$\theta_1(\varepsilon)=\varepsilon-\frac{\varepsilon^2}{2E_0^*I_0^*}(kLA_0^*G_0^*)$$

$$-\frac{\varepsilon^3}{6E_0^*I_0^*}\left\{kL(A_0^*G_1^*+A_1^*G_0^*)-\frac{kLA_0^*G_0^*}{A_0^*G_0^*}(A_0^*G_1^*+A_1^*G_0^*)\right.$$

$$\left.-\frac{2kL}{E_0^*I_0^*}(A_0^*E_1^*G_0^*I_0^*+A_0^*E_0^*G_0^*I_1^*)\right\}$$

$$-\frac{\varepsilon^4}{12E_0^*I_0^*}\left\{-\frac{3}{2}\left(\frac{I_1^*}{I_0^*}+\frac{E_1^*}{E_0^*}\right)(kL(A_0^*G_1^*+A_1^*G_0^*)-\frac{kLA_0^*G_0^*}{A_0^*G_0^*}(A_0^*G_1^*+A_1^*G_0^*)\right.$$

$$-\frac{2kL}{E_0^*I_0^*}(A_0^*E_1^*G_0^*I_0^*+A_0^*E_0^*G_0^*I_1^*)-\frac{kL}{A_0^*G_0^*}(A_0^*G_1^*+A_1^*G_0^*)^2)$$

$$+kL(A_0^*G_2^*+A_2^*G_0^*+A_1^*G_1^*)+\frac{kLA_0^*G_0^*}{2(-kA_0^*G_0^*)}\Big(2k(A_0^*G_2^*+A_2^*G_0^*+A_1^*G_1^*)$$

$$+L^2\omega^2\rho_0^*A_0^*+\frac{kL^2A_0^{*2}G_0^{*2}}{E_0^*I_0^*}-\frac{2k^2L}{kA_0^*G_0^*}(A_0^*G_1^*+A_1^*G_0^*)\Big)$$

$$\left.+\frac{kLA_0^*G_0^*}{E_0^*I_0^*}\left(\frac{kL^2}{2}A_0^*G_0^*-3E_2^*I_0^*-3E_0^*I_2^*-L^2\omega^2\rho_0^*I_0^*-3A_0^*G_0^*E_1^*I_1^*\right)\right\}+..$$

$$\theta_2(\varepsilon)=1+\frac{\varepsilon^2}{2E_0^*I_0^*}\left\{L^2(kA_0^*G_0^*-I_0^*\rho_0^*\omega^2)\right\}$$

$$-\frac{\varepsilon^3}{6E_0^*I_0^*}\left\{2L^2\left(\frac{I_1^*}{I_0^*}+\frac{E_1^*}{E_0^*}\right)(kA_0^*G_0^*-I_0^*\rho_0^*\omega^2)\right.$$

$$-kL^2(A_0^*G_1^*+A_1^*G_0^*)+L^2\omega^2(I_1^*\rho_0^*+I_0^*\rho_1^*)$$

$$\left.-\frac{A_0^*G_0^*}{kA_0^*G_0^*}L^2k^2(A_0^*G_1^*+A_1^*G_0^*)\right\}$$

$$+\frac{\varepsilon^4}{12E_0^*I_0^*}\left\{\frac{3}{2}\left(\frac{E_1^*}{E_0^*}+\frac{I_1^*}{I_0^*}\right)\left(2L^2\left(\frac{I_1^*}{I_0^*}+\frac{E_1^*}{E_0^*}\right)(A_0^*G_0^*-I_0^*\rho_0^*\omega^2)\right.\right.$$

$$\left.-kL^2(A_0^*G_1^*+A_1^*G_0^*)+L^2\omega^2(I_0^*\rho_1^*+I_1^*\rho_0^*)-\frac{A_0^*G_0^*}{kA_0^*G_0^*}L^2k(A_0^*G_1^*+A_1^*G_0^*)\right)$$

$$\left.-3\left(\frac{I_2^*}{I_0^*}+\frac{E_2^*}{E_0^*}+\frac{E_1^*I_1^*}{E_0^*I_0^*}\right)+\frac{L^2\omega^2\rho_0^*}{2E_0^*}-\frac{L^2kA_0^*G_0^*}{2E_0^*I_0^*}(L^2(kA_0^*G_0^*-I_0^*\rho_0^*\omega^2))\right\}+\ldots$$

$$\theta_3(\varepsilon)=\varepsilon-\frac{\varepsilon^2}{2E_0^*I_0^*}(E_0^*I_1^*+E_1^*I_0^*)$$

$$+\frac{\varepsilon^3}{6E_0^*I_0^*}\left\{2\left(\frac{E_1^*}{E_0^*}+\frac{I_1^*}{I_0^*}\right)(E_0^*I_1^*+E_1^*I_0^*)-2(E_0^*I_2^*+E_2^*I_0^*+E_1^*I_1^*)\right.$$

$$\left.+kL^2A_0^*G_0^*-L^2\omega^2\rho_0^*I_0^*+\frac{L^2k^2A_0^{*2}G_0^{*2}}{p-kA_0^*G_0^*}\right\}$$

$$+\frac{\varepsilon^4}{12E_0^*I_0^*}\left\{-3\left(\frac{E_0^*}{E_1^*}+\frac{I_0^*}{I_1^*}\right)(p-kA_0^*G_0^*)\right.$$

$$+\left(3\left(\frac{E_2^*}{E_0^*}+\frac{I_2^*}{I_0^*}+\frac{E_1^*I_1^*}{E_0^*I_0^*}\right)+\frac{L^2\omega^2\rho_0^*}{2E_0^*}-\frac{kL^2A_0^*G_0^*}{2E_0^*I_0^*}\right)(E_0^*I_1^*+E_1^*I_0^*)$$

$$-3(E_0^*I_3^*+E_3^*I_0^*+E_1^*I_2^*+E_2^*I_1^*)+kL^2(A_0^*G_1^*+A_1^*G_0^*)$$

$$-L^2\omega^2(\rho_0^*I_1^*+\rho_1^*I_0^*)-\frac{LA_0^*G_0^*}{2(A_0^*G_0^*)}(2kL(A_0^*G_1^*+A_1^*G_0^*)$$

$$-\frac{2kL}{A_0^*G_0^*}(A_0^*A_1^*G_0^{*2}+A_0^{*2}G_1^*G_0^*)-\frac{kLA_0^*G_0^*}{E_0^*I_0^*}(E_0^*I_1^*+E_1^*I_0^*))$$

$$\left.-\frac{L^2k}{A_0^*G_0^*}(A_0^*A_1^*G_0^{*2}+A_0^{*2}G_1^*G_0^*)\right\}+\ldots$$

References:

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