



Closed-form analytical solution procedure for element design in D regions

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Abstract:

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Keywords: reinforced concrete, membrane element, closed-form relations, nonlinear analysis, reinforcement design, D regions This paper presents a novel procedure for solving the equations system of the rotating crack model used for reinforced concrete. It is implemented in the programme NonOPt where it is used to optimise the reinforcement design of D regions. The procedure is based on solving explicit closed-form relations without the need to incrementally increase the applied loads. The solution procedure is based on a secant modulus approach and is developed initially on the basis that the stress-strain response of the steel and concrete is linearly elastic. Subsequently the effect of material nonlinearities is included and the solution procedure is adapted accordingly. A reinforcement design procedure for membrane elements is described along with some case studies. The design procedure minimises the amount of reinforcement required to satisfy predefined design constraints. Material nonlinearities are taken into account, stress and strain compatibilities are satisfied and the design considers both the ultimate and serviceability limit states through the application of appropriate design constraints.

1. Introduction

The safe, serviceable and economical design of reinforced concrete structures requires а proper determination of the reinforcement amount and distribution. To this end, it is convenient to subdivide concrete structures into B (Bernoulli) and D (Disturbed) regions. In B regions the Bernoulli's hypothesis that plane sections remain plane after loading is applicable which makes the design straightforward. However, in D regions plane sections do not remain plane owing to the geometrical or loading discontinuities. Typical examples of D regions include deep beams, pile caps, squat shear walls and beam-column connections.

D regions are typically designed using empirical design equations or strut and tie models (STM). The applicability of empirical design equations are limited and are not discussed further. Strut and tie modelling is an extension of the truss analogy used for shear in B regions. The first stage in the development of a strut and tie model involves the transformation of a continuous structure into a series of compressive concrete struts and tensile reinforcement ties. Despite the popularity and conceptual simplicity of the strut and tie method, the development of STM is not straightforward since it involves the transformation of a continuous structure into a discrete truss model (Liang et al. 2002)[12]. Even for fairly simple structural members a certain level of experience is required to decide the most appropriate STM as various models can be developed for a given structure as illustrated in Fig. 1. Further complexities arise in the estimation of the stiffness of the truss members and the evaluation of the effective concrete strength in the struts and nodes (Yun 2000[18]; Tjhin and Kuchma 2002[16]).

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Fig.1: Possible truss models for dapped-end beam: a) reinforcement layout and b & c) truss models (Fernández Ruiz et al. 2007)[8).

D regions can also be designed with linear finite element analysis which neglects the internal redistribution of stresses resulted from material nonlinearity. The authors have previously presented a nonlinear finite element based procedure for the design of reinforcement in D regions (Amini Najafian and Vollum, 2013a)[2]. The design procedure is implemented in NonOpt (Amini Najafian, 2011)[1] which is a FORTRAN program that works in conjunction with the commercial finite element program DIANA (TNO DIANA, 2007)[7]. The design procedure utilises a novel optimisation procedure which finds the minimum area of reinforcement required to satisfy the constraints subject to practical design detailing considerations. The design procedure is further verified by Amini Najafian et al. 2013[2-3-4] by designing a series of continuous beams tested experimentally by Rogowsky et al. 1986 [14]. NonOpt is developed to run more advanced design strategies (Amini Najafian and Vollum, 2013b-c)[3-4] which give slightly further optimised reinforcement designs sacrificing the procedure simplicity and computational efficiency.

A key feature of the method is that it uses the same constitutive relationships in the reinforcement design and the subsequent NLFEA for which the equations of the Modified Compression Field Theory (Collins et al., 2008)[6] are adopted. It allows explicit performance-based design constraints, such as crack widths, to be specified at the design stage. This fact and the application of general material nonlinearities including tension stiffening and strain hardening in both the analysis and design, are not considered in the surprisingly few finite element based design procedures in the literature, Fernández Ruiz and Muttoni 2007[9]; Tabatabai and Mosalam 2001[15].

This paper initially describes the equations of the Modified Compression Field Theory (MCFT). It goes on to describe the novel numerical procedure used in the design strategy to solve the equations of the MCFT with closedform relations. The procedure is novel in the sense that the equations are solved explicitly in terms of the principal compressive strain in the concrete. The solution procedure is based on a secant modulus approach which does not need the loads to be applied incrementally. The loading is assumed to be proportional and the solution procedure is developed initially on the basis that the stress-strain response of the steel and concrete is linearly elastic. Subsequently the effect of material nonlinearities is included and the solution procedure is adapted accordingly. Finally, the paper broadens the reinforcement design procedure for membrane elements, presented by Amini Najafian and Vollum 2013[2] in a wider aspect for the whole structure, in case studies on single elements. The procedure finds the minimum amount of reinforcement required to satisfy the predefined design constraints.

2. Modified Compression Field Theory

The modified compression field theory was derived from a consideration of equilibrium and strain compatibility in membrane elements (Vecchio and Collins, 1986[17]; Collins et al., 2008[6]). The theory is a rotating crack model in which cracked reinforced concrete is treated as a new material with its own stress-strain characteristics. The compatibility equations and stress-strain relationships are formulated in terms of average stresses and average strains. The values of the average stresses in the component materials differ from the local concrete and reinforcement stresses at crack locations. Therefore, it is also necessary to check local stress conditions in the reinforcement and concrete at cracks. Other key assumptions are that the principle stress directions are coincident with the principle strain directions and the concrete compressive strength is dependent on the transverse tensile strain. The theory is described by the following equations.

Equilibrium

Average stresses

$$\sigma_l = \sigma_d \cos^2 \alpha + \sigma_r \sin^2 \alpha + \rho_l f_l \tag{1}$$

$$\sigma_t = \sigma_d \sin^2 \alpha + \sigma_r \cos^2 \alpha + \rho_t f_t \tag{2}$$

$$\tau_{lt} = (-\sigma_d + \sigma_r) \sin \alpha \cos \alpha \tag{3}$$

Stresses at cracks

$$f_{lcr} = (\sigma_l + \tau_{lt} \cot \alpha - \nu_{ci} \cot \alpha) / \rho_l \tag{4}$$

$$f_{tcr} = (\sigma_t + \tau_{lt} tan\alpha + v_{ci} tan\alpha) / \rho_t$$
(5)

 σ_l , σ_t , and τ_{lt} are respectively the longitudinal, transverse and shear stresses in reinforced concrete element. σ_d (σ_r) is the compressive (tensile) principal stress in concrete, and f_l and f_t (*flor* and *ftor*) are the mean (maximum) reinforcement stresses in the two directions with reinforcement ratios of ρ_l and ρ_t . v_{cl} and α are the crack shear stress and cracking angle respectively. **Strain Compatibility**

$$\varepsilon_l = \varepsilon_d \cos^2 \alpha + \varepsilon_r \sin^2 \alpha \tag{6}$$

$$\varepsilon_t = \varepsilon_d \sin^2 \alpha + \varepsilon_r \cos^2 \alpha \tag{7}$$

$$\gamma_{lt}/2 = (-\varepsilon_d + \varepsilon_r) \sin \alpha \cos \alpha \tag{8}$$

Crack width

$$w = \varepsilon_r s_{m\alpha} \tag{9}$$

$$s_{m\alpha} = \frac{1}{\frac{|\sin\alpha|}{s_l} + \frac{|\cos\alpha|}{s_t}}$$
(10)

 ε_l , ε_t and γ_{lt} are respectively the mean longitudinal, transverse and shear strains in cracked concrete. ε_d (ε_r) is the compressive (tensile) principal strain in concrete. W is the crack width and $s_{m\alpha}$ is the average inclined crack spacing. s_l and s_t are the average crack spacings that would occur if the member were subjected to tension in the l or t direction.

$$s_l = 2c + 0.1 \frac{d_b}{\rho} \tag{11}$$

in which c is the diagonal distance to the closest l reinforcement bar in section from current depth, d_{b} is the diameter of the closest bar and ρ' stands for the steel ratio of the closest bar within a concrete area $7.5d_{b}$ above and below the bar known as the effective area. The crack spacing ^{s}t is defined similarly.

Constitutive Relationships

$$\sigma_d = \sigma_d(\varepsilon_d, \zeta) \tag{12}$$

$$\zeta = \zeta(\varepsilon_r) \tag{13}$$

$$\sigma_r = \sigma_r(\varepsilon_r) \tag{14}$$

$$f_l = f_l(\varepsilon_l) \tag{15}$$

$$f_t = f_t(\varepsilon_t) \tag{16}$$

where ζ is a softening coefficient which depends on the principal tensile strain in the concrete, ε_r , as proposed by

Vecchio and Collins (1986). Fig. 2 depicts the constitutive relations for cracked concrete.

Shear stresses on crack

$$v_{cimax} = \frac{0.18 \sqrt{f_c'}}{0.31 + 24w/(a+16)} \tag{17}$$

in which f_c' is the concrete compressive strength in MPa, a is the maximum aggregate size in mm and v_{cimax} is the maximum local shear stress in MPa that crack is able to transfer.



Fig.2: Stress-strain relationships for cracked concrete

3. Solution Procedure

The eleven governing equations for a membrane element, equations (1) to (3), (6) to (8) and (12) to (16) contain 14 unknown variables (seven stresses, σ_l , σ_t , τ_{lt} , σ_d , σ_r , f_l and f_t , five strains, ε_l , ε_t , γ_{lt} , ε_d , ε_r , the cracking angle, α , and the softening coefficient, ζ). When three unknown variables e.g. applied stresses σ_l , σ_t and τ_{lt} are given, the remaining 11 unknowns can be found by solving the eleven governing equations.

The authors have developed a novel solution procedure for solving the 11 equations of the MCFT, in terms of the applied stresses, which is believed to be computationally efficient. The procedure is utilises a secant modulus approach in which the loads are applied in a single step. The solution procedure is initially formulated assuming that the stress-strain response of the steel and concrete is linearly elastic and σ_r equals zero. In this case the two unknown variables, \mathcal{E}_l and \mathcal{E}_t , in (1) and (2) are substituted with their values from the strain compatibility equations, (6) and (7), so we have:

$$\sigma_l = (E_c + \rho_l E_{sl}) \varepsilon_d \cos^2 \alpha + \rho_l E_{sl} \varepsilon_r \sin^2 \alpha \quad (18)$$

$$\sigma_t = (E_c + \rho_t E_{st}) \varepsilon_d \sin^2 \alpha + \rho_t E_{st} \varepsilon_r \cos^2 \alpha \quad (19)$$

where E_c , E_{sl} and E_{st} are elastic moduli of concrete and reinforcements in the longitudinal and transverse directions respectively. Multiplying (18) and (19) respectively by $\rho_t E_{st} \cos^2 \alpha$ and $\rho_l E_{sl} \sin^2 \alpha$ and subtracting the results removes the unknown, ε_r .

$$\rho_{l}\rho_{t}E_{sl}E_{st}\varepsilon_{d}\cos 2\alpha + E_{c}\varepsilon_{d}(\rho_{t}E_{st}\cos^{4}\alpha - \rho_{l}E_{sl}\sin^{4}\alpha) +\rho_{l}E_{sl}\sigma_{t}\sin^{2}\alpha - \rho_{t}E_{st}\sigma_{l}\cos^{2}\alpha = 0$$
(20)

In addition, α can be determined by rearranging (3) as follows:

$$\alpha = \frac{1}{2} \sin^{-1} \left(\frac{-2\tau_{lt}}{E_c \varepsilon_d} \right)_{,} - \frac{\pi}{4} \le \alpha \le \frac{\pi}{4}$$
(21)

$$\alpha = \frac{\pi}{2} - \frac{1}{2} \sin^{-1} \left(\frac{-2\tau_{lt}}{E_c \varepsilon_d} \right), \frac{\pi}{4} \le \alpha \le \frac{3\pi}{4}$$
(22)

Note that $\varepsilon_d \neq 0$ as in the rotating crack model it is assumed that $\tau_{lt} \neq 0$ (see equation 3). Substituting α from (21) and (22) as appropriate into (20), allows the equations of the MCFT to be reduced to the following two closed-form equations in which the only unknown is ε_d .

$$\begin{split} f_{1}(\varepsilon_{d}) &= ((\rho_{l}\rho_{t}E_{sl}E_{st} + (\rho_{l}E_{sl} + \rho_{t}E_{st})\frac{E_{c}}{2})\varepsilon_{d} \\ &- \frac{\rho_{l}E_{sl}\sigma_{t} + \rho_{t}E_{st}\sigma_{l}}{2})\sqrt{1 - \frac{4\tau_{lt}^{2}}{E_{c}^{2}\varepsilon_{d}^{2}}} \\ &+ (\rho_{t}E_{st} - \rho_{l}E_{sl})(\frac{E_{c}\varepsilon_{d}}{2} - \frac{\tau_{lt}^{2}}{E_{c}\varepsilon_{d}}) \\ &+ \frac{\rho_{l}E_{sl}\sigma_{t} - \rho_{t}E_{st}\sigma_{l}}{2} = 0, \qquad -\frac{\pi}{4} \leq \alpha \leq \frac{\pi}{4} \quad (23) \\ f_{2}(\varepsilon_{d}) &= (-(\rho_{l}\rho_{t}E_{sl}E_{st} + (\rho_{l}E_{sl} + \rho_{t}E_{st})\frac{E_{c}}{2})\varepsilon_{d} \\ &+ \frac{\rho_{l}E_{sl}\sigma_{t} + \rho_{t}E_{st}\sigma_{l}}{2})\sqrt{1 - \frac{4\tau_{lt}^{2}}{E_{c}^{2}\varepsilon_{d}^{2}}} \\ &+ (\rho_{t}E_{st} - \rho_{l}E_{sl})(\frac{E_{c}\varepsilon_{d}}{2} - \frac{\tau_{lt}^{2}}{E_{c}\varepsilon_{d}}) \\ &+ \frac{\rho_{l}E_{sl}\sigma_{t} - \rho_{t}E_{st}\sigma_{l}}{2} = 0, \qquad \frac{\pi}{4} \leq \alpha \leq \frac{3\pi}{4} \quad (24) \end{split}$$

Equations (23) and (24) are differentiable functions (see Fig. 3) and as \mathcal{E}_d must be negative, multiple extra solutions are omitted by applying a negative solution domain for \mathcal{E}_d . Bracketing methods, such as Bisection Method or the False-position Method, are most appropriate for finding the solution as open methods, such as the Newton-Raphson Method, can lead to premature termination of computations as the tangent line to the curve in which there is the solution at some point can intersect the \mathcal{E}_d axis outside the valid domain as depicted in Fig. 3. Having found \mathcal{E}_d , the remaining unknowns can be calculated directly.



Fig. 3: Solution in the closed-form relations: a) open methods diverge on the right side of the solution and b) open methods diverge on the left side of the solution.

3.1 Nonlinear concrete.

This procedure is readily extended to allow for the effect of concrete material nonlinearities in compression by updating the secant modulus until the stress obtained from the secant elastic modulus is adequately equal to the stress calculated with the specified nonlinear stress-strain relationship (see Fig. 4). Further details of the solution procedure can be found in Amini Najafian, 2011[1].



Fig.4: Secant modulus computational process: a) there is a solution and b) there is no solution (overstressed element).

Tension stiffening is taken into account by modifying the applied stresses by an amount equivalent to the stress resisted by concrete in tension. Normal equivalent stresses

stress shear equivalent defined and are as $\sigma_{eq,l} = \sigma_l - \sigma_r \sin^2 \alpha_{, \sigma_{eq,t}} = \sigma_t - \sigma_r \cos^2 \alpha_{\text{and}}$ $\tau_{eq,lt} = \tau_{lt} - \sigma_r sin\alpha \cos\alpha$. The next stage is to express the three equations of equilibrium, i.e. (1) to (3), in terms of the equivalent stresses. This allows the same solution procedure to be used as previously. The only change is that the equivalent stresses are unknown at the beginning of the analysis. Therefore, an iterative analysis is needed to find the two unknowns, α and ε_r , which define $\sigma_{eq,l}, \sigma_{eq,t}, and \tau_{eq,lt}$



Fig.5: Tension stiffening computations.

Fig.5 illustrates the procedure used to calculate the equivalent stresses. The equivalent stresses are calculated in a nested loop in which the concrete is assumed to be linear in compression for reasons of numerical stability. aim is to determine $\alpha^{new} = \alpha^{old}$ The and $\varepsilon_r^{new} = \varepsilon_r^{old}$ within acceptable tolerances where ε_r^{old} and α^{old} are the values from the previous iteration which are used to obtain the updated values, ε_r^{new} and α^{new} The unknowns, ε_r and α , are found sequentially as shown in Figure 5. The figure shows that the cracking angle is found for each value of $\varepsilon_r (\alpha^{new} = \alpha^{old})$, then the equivalent stresses from the alpha and ε_r^{old} are calculated and the second loop continues until \mathcal{E}_r converges. The procedure starts by finding the equivalent stresses in a loop for which the system of equations is solved with linear concrete in compression. The computations continue until the equivalent stresses converge after which the secant modulus of the concrete is adjusted in the outer loop until

the stress obtained from the secant elastic modulus is adequately equal to that calculated with the specified nonlinear stress-strain relationship.

4. Reinforcement Optimisation

This section describes the procedure developed to optimise the reinforcement design in a single element. Similar procedure is used by Amini Najafian and Vollum (2013a)[2] for design of groups of elements in the structure. The design procedure minimises the sum of the reinforcement ratios in the element subject to predefined design constraints. The design considers both the ultimate (ULS) and serviceability limit states (SLS) through the application of appropriate constraints. The constraints and their associated factor of safeties are expressed in terms of mean strains in the component materials or maximum stresses in the reinforcements at cracks. Applying the design constraints, prevents element failure, ensures the safe transmission of the applied loads through the cracks, limits the deformations to prescribed limits and ensures practical values for the reinforcement ratios. The design satisfies both stress and strain compatibilities, and takes account of the tensile capacity of concrete, concrete nonlinearity and the post-yield strength of reinforcing bars.

4.1. General Concept

The total weight of reinforcement in a rectangular element with sides of length of l_l and l_t , equals $(\rho_l + \rho_t) t l_l l_t$ where t is the element thickness. Therefore, the reinforcement weight is minimised by finding the minimum sum of the reinforcement ratios ρ_l and P_t . The final stress and strain states inside the element are dependent on the reinforcement ratios in both the l and t directions. It follows that the values of the required reinforcement ratios in the l and t directions are interdependent and there is no unique solution for ρ_1 and ρ_t . The aim of the design is to find the minimum sum of $\rho_l + \rho_t$ which satisfies the design constraints by solving the 11 equations of the rotating crack model. The knowns in the system of equations are the applied stresses, σ_l , σ_t and τ_{lt} , as well as the material properties. The unknowns are the internal strains, \mathcal{E}_d , \mathcal{E}_r , \mathcal{E}_l , \mathcal{E}_t and \mathcal{Y}_{lt} , the stresses in the reinforcements, f_l and f_t , the stresses in the concrete, σ_d and σ_r , the cracking angle, α , and the reinforcement ratios, ρ_l and ρ_t . The unknown shear strain, γ_{lt} , only appears in equation 8 and can be calculated directly in terms of the other unknowns.

A number of optimisation methods are available in the literature but there is no known method for determining the minimum solution of a general problem (Ozgur 2005)[13]. According to comparative studies (Kao 1998)[11], the generalised reduced gradient methods and the sequential quadratic programming methods are two of the best deterministic local optimisation methods. These methods require gradient information unlike methods such as genetic algorithms, simulated annealing and tabu search which can be used for non-differentiable discontinuous highly nonlinear objective constrained functions. General optimisation methods may be used to solve the reinforcement minimisation problem when linear concrete and linear reinforcement are applied, but they cannot be used for nonlinear material properties due to the presence of extra solutions as explained by Amini Najafian (2011). These extra solutions mislead general optimisation methods which can generate spurious solutions having no physical meaning.

4.2. Proposed Solution Procedure

The minimum area of reinforcement is obtained by generating an equally spaced mesh in the $\rho_l - \rho_t$ plane (see Fig. 6) between the minimum and maximum permitted reinforcement ratios, ρ_{min} and ρ_{max} . The solution procedure involves moving through the mesh on parallel lines from the minimum to the maximum value of $\rho_l + \rho_t$. At each coordinate the design constraints are checked until all the conditions are fulfilled. This involves first solving the rotating crack equations and then checking the design constraints. When a solution is obtained in the $\rho_l - \rho_t$ plane, the mesh is refined around the solution and the computations are repeated on the refined mesh until the required level of precision is achieved. In the $\rho_l - \rho_t$ plane, the sum of reinforcement ratios at each point is related to the perpendicular distance from the point to the line Λ_{\min} :

$$\Lambda_{\min}: \rho_l + \rho_t = 2\rho_{\min} \tag{25}$$

$$dis(n_i, \Lambda) = (\rho_{li} + \rho_{ti} - 2\rho_{min})/\sqrt{2}$$
⁽²⁶⁾

It is clear that $\rho_l + \rho_t = const$ defines a set of parallel lines in the $\rho_l - \rho_t$ plane between Λ_{\min} and Λ_{\max} . The total amount of reinforcement, $\rho_{total} = \rho_{li} + \rho_{ti}$, is constant along each line and increases from $2\rho_{min}$ to $2\rho_{max}$ as the line moves from $\Lambda_{\min \text{ towards}} \Lambda_{\max}$ (see Fig. 6).



Fig.6: Mesh generation and refinement

When the amount of reinforcement in a coordinate is recognised to be enough to fulfil the design constraints, the mesh is refined between the parallel line passing through the solution and the adjacent parallel line with δ_{ρ} less total reinforcement. The incremental step $\delta_{\rho} = (\rho_{max} - \rho_{min})/n_{d}$ in which n_{d} is the number of mesh divisions applied by the user. The mesh refinement includes both the bracketing parallel lines. The incremental $\frac{\delta_{\rho}}{\rho}$

step is taken as n_d in the refined mesh.

The mesh refinement may need to be repeated several times before the required precision is achieved. Fig. 6 shows a mesh generated for a case with $n_d = 4$ and $n_r = 1$ where n_r is the number of mesh refinements. The mesh is refined once around points A and B which correspond to the initial solutions for two different loading conditions. In the first loading condition, a point on Λ_A satisfies all the design constraints and therefore all the points on lines, Λ_{A1} , Λ_{A2} , Λ_{A3} , Λ_{A4} and Λ_A are checked sequentially to find a solution. Similarly, for the second loading condition, the valid answer is on Λ_B and so the coordinates on lines Λ_{B1} , Λ_{B2} , Λ_{B3} , Λ_{B4} and Λ_B

Note that when a point on the final refined mesh fulfils all the constraints, the computations continue along that current parallel line as there may be more than one point on the line which satisfies all the design constraints. The final solution is taken as that with the greatest factor of safety in cases where multiple solutions exist with the same value of

 ρ_{total} . The overall factor of safety of an element is

defined as the least of the factors of safety calculated for each design constraint as described in section 4.3. The flowchart in Fig. 7 illustrates the procedure used to find the minimum area of reinforcement.



Fig.7: Flowchart for finding the minimum reinforcement design in an element.

4.3 Design Constraints

The final reinforcement ratios depend on the design constraints adopted for the serviceability and ultimate limit states. The maximum load that can be carried by a membrane element is limited by the ultimate compressive strength of the concrete or the tensile strength of the reinforcement. Deformations are controlled by limiting the principal tensile strain in concrete as well as the reinforcement strains in the l and t directions. The minimum and maximum reinforcement ratios are also limited in accordance with structural codes. Consequently, the design constraints are expressed in terms of maximum and minimum reinforcement ratios, maximum permissible stresses in the reinforcement at cracks, and mean strains in both the concrete and reinforcement. Limiting the mean strains in the concrete controls both the maximum crack width and the concrete compressive stress which depends on both the principal compressive and tensile strains.

The factor of safety for the average strains is defined as the ratio of the permissible strain to the actual strain. For instance, the factor of safety for tensile principal strain in rdirection is calculated as ε_r , where $\varepsilon_{r,per}$ is the permissible strain. A different approach is used to calculate the safety factor for the reinforcement at cracks since the exact values of maximum stresses are unknown due to indeterminacy. The safety factor at the crack is defined as $SF = Min\left(\left|\frac{f_{slcr,per}}{f_{slcr}}\right|, \left|\frac{f_{stcr,per}}{f_{stcr}}\right|\right) \text{ when } \boldsymbol{v}_{ci} = \boldsymbol{0} \text{ or the reinforcement is overstressed in both directions.}$

 $SF = \frac{v_{cl,per}}{v_{cl,per}}$

 $v_{ci} \neq 0$ and the least stressed reinforcement is not overstressed as discussed by Amini Najafian (2011)[1].

5. Case Studies

Two case studies are presented in this section to compare the design results of membrane elements in cases with linear and nonlinear material properties. The elastic moduli of $E_c = 2.22E4 MPa$ and $E_s = 2E5 MPa$ are considered for the linear case and the Hognestad parabola (Hognestad, 1951)[10] with compressive strength $f_c' = 25 MPa$ and strain at peak stress $\varepsilon_c' = -0.0022$ is used for nonlinear modelling of concrete.



Fig.8: Strain variation through the initial mesh in Case study 1 – Load case 1: a) concrete compressive strain, b) strain in the lbars and c) strain in the t-bars.(Continued)



Fig.8: (Continued)



Fig.9: Variations of reinforcement strains through the refined mesh in Case study 1 – Load case 1: a) strain in the l-bars and b) strain in the t-bars.

The design constraints limit the average tensile strains in the reinforcement to $\varepsilon_{l,per} = \varepsilon_{t,per} = 0.0025$ and the compressive strain in the concrete to $\varepsilon_{d,per} = -0.002$. The minimum and maximum permitted reinforcement ratios are assumed to be 0.004 and 0.04. The number of mesh divisions is taken as $n_d = 10$ and the initial mesh $(n_r = 0)$ is refined with one level of refinement $(n_r = 1)$ unless otherwise stated.

5.1. Linear material properties (Case study 1)

This study considers a membrane element with linear material properties. The element is initially subjected to Load case 1 with equal normal stresses $\sigma_l = \sigma_t = 4 MPa_{and shear stress} \tau_{lt} = 5 MPa_{and shear stress}$ variations in the concrete compressive strain and reinforcement strains in the first generated mesh are shown in Figure 8 in which compressive strains are negative. The compressive strain in the concrete is distributed over the mesh symmetrically since $\sigma_l = \sigma_t$ and the material

properties are the same for the *l* and *t* reinforcements. Figure 8 (a) shows that the compressive strains in concrete are not critical in this case as the peak strain is less than the limiting value of -0.002. However, the tensile stresses in the steel bars are only less than the permitted stress in the dark blue area in the contours in Figures 8 (b) and (c). From these contours it is seen that the minimum required reinforcement ratios for this symmetrical loading case are $\rho_l = \rho_t = 0.01840$ where the strains in the steel bars equal 2.4456E - 3 in the two directions. It is clearly unnecessary to analyse all the coordinates in the mesh as is done in this example for purpose of illustration. In reality, the solution procedure can be stopped as soon as a solution is found. Figure 9 shows the refined mesh ($n_r = 1$) adjacent to the initial solution and be seen that the minimum reinforcement obtained after the mesh refinement is $\rho_l = \rho_t = 0.01804$. Table 1 gives the analysis results for these two solutions.

Load case	n _r	ρι	ρ _t	ε _d	ει	ε _t	a (rad)	Ptotal
1	0	0.018400	0.018400	-4.5045E-4	2.4456E-3	2.4456E-3	0.7854	0.036800
	1	0.018040	0.018040	-4.5045E-4	2.4945E-3	2.4945E-3	0.7854	0.036080
2	0	0.011200	0.018400	-1.9630E-3	2.4917E-3	-1.9028E-3	1.4550	0.029600
	1	0.011200	0.016240	-1.9948E-3	2.4875E-3	-1.9361E-3	1.4569	0.027440
	2	0.011164	0.015952	-1.9991E-3	2.4949E-3	-1.9405E-3	1.4572	0.027116
		0.011200	0.015916	-1.9996E-3	2.4868E-3	-1.9412E-3	1.4572	0.027116

Table.1: Solutions in Case study 1.



Fig.10: Strain variation through the initial mesh in Case study 1 – Load case 2: a) concrete compressive strain, b) strain in the 1-bars and c) strain in the t-bars.(Continued)

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Fig.10: (Continued)

The applied loads are then increased to make the concrete become critical in compression (Load case 2). The revised loadings are $\sigma_l = 5MPa$, $\sigma_t = -50MPa$ and $\tau_{lt} = 5MPa$. The design constraints and number of mesh divisions are the same as before but two levels of mesh refinement are applied.

The contour in Figure 10 (c) shows that the transverse strains are no longer critical anywhere in the $\rho_l - \rho_t$ plane. However, the longitudinal strains as well as the concrete compressive strains are acceptable only in the dark blue areas of Figures 10 (a) and (b) from which the optimum reinforcement ratios given in Table 1, $\rho_l = 0.0112$ and $\rho_t = 0.0184$, are determined. The values of the strains for the solution in the initial mesh are given in Table 1 show the solutions after the first and the second mesh refinements. In the third generated mesh, the second mesh refinement, two coordinates satisfy all the design constraints and are valid solutions.

5.2. Influence of Nonlinear Concrete (Case study 2)

This example redesigns the element considered in Case study 1 with nonlinear concrete. The Hognestad parabola is used to define compressive stress-strain relationship for concrete. The reinforcement is assumed to be linearly elastic as before. The element is initially subjected to Load case 1 from Case study 1 where $\sigma_l = \sigma_t = 4 MPa$ and

$$\tau_{lt} = 5 MPa$$

Table 2 shows the outcome of the design procedure for load case 1 with no mesh refinement, $n_r = 0$, and for one level of mesh refinement, $n_r = 1$. The minimum reinforcements are coincidentally equal to those which were derived assuming that the concrete was linearly elastic in compression (see Table 2). The cracking angle and the strains are also similar except for \mathcal{E}_d . In this example, the compressive principal strain in concrete is very small and as a result the effect of concrete nonlinearity is insignificant. Figure 11 depicts the strain distributions over the first generated mesh. The permitted strain contours for the steel bars are clearly labelled with the value of 0.0025 to enable the minimum possible reinforcement to be verified from the figure. For some combinations of \mathcal{P}_l and

Pt, the membrane element is instable and therefore the strain distributions do not cover the entire mesh. The symmetrical form of the contours, due to the symmetrical loading and reinforcement properties, is seen in this figure.

To compare the design results given from the element modelled with linear concrete with those for the element modelled with nonlinear concrete, the second loading condition in Case study 1, $\tau_{lt} = 5 MPa$, $\sigma_l = 5 MPa$ and $\sigma_t = -50 MPa$, was applied but in this loading condition the element could not carry the loads with any reinforcement combination in the permitted range. Therefore, the stresses were reduced to $\tau_{lt} = 3 MPa$, $\sigma_l = 3 MPa$ and $\sigma_t = -30 MPa$ in Load case 3. The results for this loading condition are given in Table 2 for $n_r = 0$ for both linear $(E_c = 2.22E4MPa)$ and nonlinear concrete. In this case, the compressive strain is critical for the nonlinear concrete as shown in Table 2.

The strains in the nonlinear case are plotted over the initial mesh in Figure 12 from which the minimum required reinforcements in Table 4 can be checked. The strains in the transverse direction are always compressive and they are in the permitted domain so long as the element is stable. However, the longitudinal strains exceed the permitted strain beyond the -0.0025 contour as shown in Figure 12 (b). The concrete compressive strains also violate the permitted strain at some coordinates. The limit for this strain is -0.002 which separates clearly the invalid area, the blue area in Figure 12 (a), from the acceptable area.

Table.2:	Solution	ns in (Case	study 2	2.
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Load case	concrete	n _r	ρι	ρ _t	ε _d	ει	ε	α (rad)	σ _d (MPa)
1	nonlinear	0	0.01840	0.01840	-9.5854E-4	2.4457E-3	2.4457E-3	0.7854	-10.0000
		1	0.01804	0.01804	-9.7198E-4	2.4945E-3	2.4945E-3	0.7854	-10.0000
3	linear	onlinear 0	0.0076	0.0040	-1.3192E-3	2.1780E-3	-1.2817E-3	1.4676	-29.2853
	nonlinear		0.0112	0.0184	-1.9112E-3	1.5126E-3	-1.8538E-3	1.4421	-23.5663
			0.0148	0.0148	-1.9606E-3	1.1385E-3	-1.9135E-3	1.4481	-24.7058



Fig.11: Strain variation through the mesh in Case study 2 – Load case 1: a) concrete compressive strain, b) strain in the l-bars and c) strain in the t-bars.(Continued)









Fig.12: Strain variation through the mesh in Case study 2 – Load case 3: a) concrete compressive strain, b) strain in the l-bars and c) strain in the t-bars.(Continued)



Fig.12: (Continued)

6. Concluding Remarks

This paper proposes a novel technique for solving the equations of the rotating crack model in the MCFT. The solution procedure is accelerated by expressing the equations in closed-form. The analysis results are obtained from the final values of the stresses without the need for incremental loading steps. The equations for the rotating crack model are expressed in terms of one unknown, compressive strain in concrete, assuming linear elastic behaviour for the concrete and reinforcement.

A design procedure is presented which minimises the area of reinforcement required to satisfy the design constraints inside a membrane element. The strain compatibility equations are satisfied as well as the stress compatibility relations. The design takes account of the effects of tension stiffening and the nonlinear stress-strain response of steel and concrete. The design constraints are used to limit the deformations and stresses inside the element. It allows the designer to use the procedure for both the serviceability and ultimate limit states. To avoid concrete crushing, the average compressive stress in concrete is limited in terms of its principal compressive strain to avoid issues arising from softening. Crack widths are limited by controlling the maximum principal tensile strain in the concrete. In addition, there are limits on the average strains and maximum stresses in the reinforcing bars. For a practical design the values of reinforcement ratios should be in the valid domain proposed in structural codes, which is included in the proposed procedure as well. The design procedure has been verified successfully for various loading conditions and material properties. Data visualisation of the results validates the performance of the procedure.

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