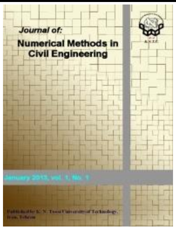

Numerical Methods in Civil Engineering

Assessment of bending solution of beam with arbitrary boundary conditions: an accurate comparison of various approaches

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Abstract:

Bending responses are the important characteristics of structures. In this paper, the bending solution of the thin and thick beams which are elastically restrained against rotation and translation are presented using various theories. Hence, accurate and direct modeling technique is offered for modeling of the thin and thick beams. The effect of the values of the span-to-depth ratio and type of the beam supports are assessed to state accurate comparison of various theories. Finally, the numerical examples are shown in order to present the evaluation of the efficiency and simplicity of the various theories. The results of the theories are compared with the results of the finite element method (ABAQUS). Based on the results, using the Timoshenko beam theory, the obtained values are in good agreement with the Finite Element modeling for the values of the span-to-depth ratio (L/h) less than 3. On the other hands, due to ignoring the shear deformation effect, the Euler–Bernoulli theory underestimates the deflection of the moderately deep beams (L/h=5).

1. Introduction

Beams are widely used as classical structural components in structural engineering applications. Owing to their practical importance, much effort has been devoted to the static and dynamic analysis of these structural components. Hence, the beam theory has been and is still a subject which has been extensively studied for a century. In the theory of beams, two different limit cases are usually considered: the Euler-Bernoulli beam theory and the Timoshenko beam theory. In the context of the beam theory, the simplest one is the Euler-Bernoulli (classical thin) beam theory (EBT) which overlooks the shear deformation in the beam thickness. Therefore, the shear strains and the shear stresses are eliminated from this theory. However, the thick and the moderately thick beams are characterized by the non-negligible shear deformations in the thickness since the longitudinal elastic modulus is much higher than the shear and the transversal module. Hence, the use of a shear deformation beam theory is recommended. The Timoshenko model is known as first-order shear deformation theory (FSDT) and accounts for the shear deformation in thickness in the simplest way (Timoshenko, 1921) [1]. This model provides satisfactory results for a wide class of structural problems, including the moderately thick beam. Moreover, it is used in the large-scale computations typical of the industrial applications due to its computational efficiency. A beam is typically considered to be thin when the ratio of its thickness to length is 1/20 or less. In fact, some of the beams used in the practical applications satisfy this criterion. Thus, this usually permits the use of classical thin beam theory to obtain the beam behavior with good accuracy. However, the structural behavior determined by thin beam theory will not be accurate for the beam with a thickness ratio of 1/20. These inaccuracies are largely eliminated by use of the FSDT, as it does include the effects of the additional beam flexibility due to the shear deformation, and the additional beam inertia due to the rotations (supplementing the translational inertia). Both effects decrease the natural frequency and increase the deflection and the critical buckling load of the beam. On the other hand, there are still other effects not accounted for by the FSDT (e.g., the warping of the normals to the mid-plane, the stretching in the thickness direction), but these are typically unimportant mainly for the vibration, bending, and buckling problems up until the very thick beams are encountered.

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Therefore, a three-dimensional analysis should be used for the very thick beams. The third-order shear deformation theory (TSDT) is proposed for the beams with rectangular cross-section by Wang et al. (2000) [2]. The parabolic distribution of the transverse shear stress and strain with respect to the thickness coordinate are assumed in the TSDT. On the other hands, the zero transverse shear stress condition of the upper and lower fibers of the cross section is satisfied without a shear correction factor in the TSDT (Simsek and Kocaturk, 2007) [3]. Ghugal and Sharma (2009) applied the hyperbolic shear deformation theory for the static and dynamic analysis of the thick beam [4]. The parabolic, trigonometric, hyperbolic and exponential functions are used in terms of thickness to represent the effect of the transverse shear deformation by Sayyad (2011) [5]. Sayyad and Ghugal (2011) [6] developed new hyperbolic shear deformation theory for the thick beam flexure, in which combined effect of the shear and the bending rotations is considered. The new Trigonometric shear deformation theory is developed for the bending of the thick beam by Naik et al. (2012) [7]. Three models for a cantilever beam based on the Euler–Bernoulli, Timoshenko and two-dimensional elasticity is presented by Labuschagne et al. (2009) [8]. The results showed that the Timoshenko beam theory is close to the two-dimensional theory for modes of practical importance, whereas, the applicability of the Euler–Bernoulli beam theory is limited. Yavari et al. (2000) presented some applications of the distribution theory of Schwarz to the beam bending problems [9]. The steady-state response of the quadratic nonlinear oscillator under the weak and the strong external excitations is offered by Jiang et al. (2015) [10]. The Lindstedt-Poincare method, the multiple-scale method, the averaging method, and the harmonic balance method is applied for the analytical approximations of the amplitude frequency response. The single variable beam theories taking into account effect of transverse shear deformation for the buckling, bending and free vibration analysis of the thick beam are presented by Sayyad and Ghugal (2016) [11]. Also, Thai and Vo (2012) developed the various higher order shear deformation beam theories for the free vibration and bending of functionally graded beams [12]. A new trigonometric shear deformation theory for the composite sandwich and the laminated plates is developed by Mantari et al. (2012) [13]. Solutions for the free vibration of beams with solid and thin-walled cross-sections are supplied using the refined theories based on the displacement variables by Dan et al. (2016) [14]. Petrolo et al. (2016) presented the free vibration analysis of the damaged beams by means of the 1D (beam) advanced finite element models [15]. Also, Cinefra et al. (2017) presented the best theory diagrams (BTDs) for multilayered plates involved in multi-field problems [16]. A thin-walled beam with a varying quadrilateral cross-section is formulated based on the higher order beam theory by Choi et al. (2017) [17]. Wang (1995) offered the deflection and the stress resultants of the Timoshenko beam in terms of the Euler-Bernoulli beam solutions [18]. The transverse vibration of a Timoshenko beam with the one-step change in cross-section subjected to an axial force is developed by Janevski et al. (2014) [19]. In the present study, the bending solution using analytical method is introduced for the analysis of the Euler–Bernoulli and the Timoshenko beams with arbitrary boundary conditions. Furthermore, the analysis for the Euler–Bernoulli and Timoshenko beams are written in a general form. In general, the main objective of this paper is to state the accuracy of various theories for analysis of the beam with the various span-to-depth ratio (L/h). For this purpose, numerical examples are presented in order to evaluate the efficiency and simplicity of the various theories. Furthermore, the results of the theories are compared with the results of the finite element method (ABAQUS).

This paper is organized as follows. Section 2 and 3 outline the basic equations in detail for arbitrary boundary conditions, based on the Euler–Bernoulli beam and the Timoshenko beam theories. Susequently, relationships between the Euler-Bernoulli beam and the Timoshenko beam are presented in section 4, wheras, section 5 presents some numerical examples to illustrate the efficiency of the various methods. Finally, in section 6, conclusions are drawn, briefly.

2. Euler-Bernoulli Beam Theory

The simplest beam theory is known as the Euler-Bernoulli beam theory (EBT) or simply the classical beam theory (CBT). The EBT is based on the following assumptions known as the Euler-Bernoulli assumptions (Carrera et. al., 2011) [20]:

- The cross-sections of the beam do not deform in a significant manner under the application of transverse or axial loads and can be assumed as rigid
- During deformation, the cross section of the beam is assumed to remain planar and normal to the deformed axis of the beam.

These assumptions have been confirmed on a large scale for slender beams made of isotropic materials with solid cross-sections. These assumptions allow to describe the plate deformation in terms of certain displacement quantities. A beam element with length dx between two cross-sections

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taken normal to the deflected axis of the beam is shown in Fig. 1 (Reddy, 1993) [21]. The first Euler-Bernoulli assumption requires that the displacement field be a linear function of the thickness coordinate z:

$$u(x,z) = u_0 + z \cdot F_1(x) \tag{1}$$

$$w(x,z) = w_0 + z \cdot F_2(x) \tag{2}$$

where (u_0; w_0; F_1; F_2) are functions to be determined such that, the remaining two assumptions of the Euler-Bernoulli hypothesis are satisfied. The inextensibility assumption requires that

$$\frac{\partial w}{\partial z} = 0 \quad \rightarrow \quad F_2 = 0 \quad \textit{for all } x \tag{3}$$

where w is independent of z, i.e., w = $w_0(x)$. On the other hand, Euler-Bernoulli assumption requires that

$$\frac{\partial u}{\partial z} = -\frac{\partial w}{\partial x} \quad \rightarrow \quad F_1 = -\frac{\partial w_0}{\partial x} \tag{4}$$

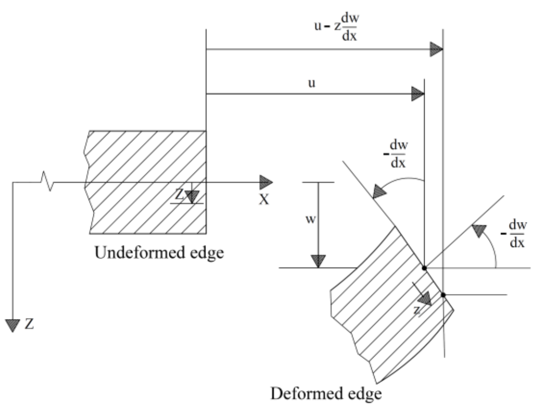

Fig.1: The coordinate system and notation for the Euler-Bernoulli beam

Hence, the displacement field (1-2) takes the form

$$u(x,z) = u_0 - z \cdot \frac{\partial w_0}{\partial x} \tag{5}$$

$$w(x,z) = w_0(x) \tag{6}$$

Thus, the Euler-Bernoulli beam deformation is completely determined by the functions (u_0; w_0), which denote the displacements of a point on the mid-plane along the two coordinate directions. Note that the displacement field (5-6) will result, in spite of neglecting all transverse strains and this will be presented in the following pages of the paper. According to the Euler-Bernoulli beam theory, the governing differential equation for the considered beam can be given by:

$$\frac{d^2}{dx^2}\left(EI \frac{d^2 w^E}{dx^2} \right) = q \tag{7}$$

where w^E is the transverse displacement of the mid-surface of the beam in the z-direction and q presents the external load on the beam which is an arbitrary function of coordinate x. In addition, I and E are the second moment of area and Young's modulus of elasticity, respectively. The superscript E denotes the Euler-Bernoulli beam quantities. Based on Hooke's law and the Euler-Bernoulli's assumptions, the bending moment–displacement and the shear force–displacement relations are given by:

$$M^E = -EI \frac{d^2 w^E}{dx^2} \tag{8}$$

$$V^E = -\frac{d}{dx}\left(EI \frac{d^2 w^E}{dx^2} \right) \tag{9}$$

Thin beam bending solutions can be obtained by solving the foregoing governing equation together with the natural boundary conditions.

Although the classical fourth-order beam theory of Euler-Bernoulli, has been a very useful engineering approximation, it has some drawbacks. This theory neglects the shear deformations, and as a result, it underestimates deflections and overestimates stresses.

3. Timoshenko Beam Theory

Over the years, researchers have tried to modify the classical beam theory to relax its restrictions. Several alternative beam theories have appeared in the literature, among which those of Timoshenko's is the most well known .The theory is based on the following assumptions:

- The cross-section is rigid and constant throughout the length of the beam and has one plane of symmetry
- Shear deformations of the cross-section of the beam are taken into account while the elastic axial deformations are ignored

A Timoshenko beam element with length dx between two cross-sections taken normal to the deflected axis of the beam is shown in Fig. 2. According to the first-order shear deformation theory, the displacement field is a linear function of the thickness coordinate z:

$$u(x,z) = u_0 + z \cdot \phi_1(x) \tag{10}$$

$$w(x,z) = w_0 + z \cdot \phi_2(x) \tag{11}$$

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where (u_0; w_0; F_1; F_2) are functions to be determined such that the remaining two assumptions of the Timoshenko hypothesis are satisfied. The inextensibility assumption requires that:

$$\frac{\partial w}{\partial z}=0 \quad \rightarrow \quad \phi_2=0 \qquad \textit{for all } x \tag{12}$$

Thus, w is independent of z, i.e., w = $w_0(x)$. Timoshenko assumption requires that

$$\frac{\partial u}{\partial z}=-\psi(x) \quad \rightarrow \quad \phi_1=-\psi(x) \tag{13}$$

Finally, the displacement fields are given by

$$u(x,z)=u_0-z\cdot\psi(x) \tag{14}$$

$$w(x,z)=w_0(x) \tag{15}$$

Thus, the Timoshenko beam deformation is completely determined by the functions (u_0; w_0), which denote the displacements of a point on the mid-

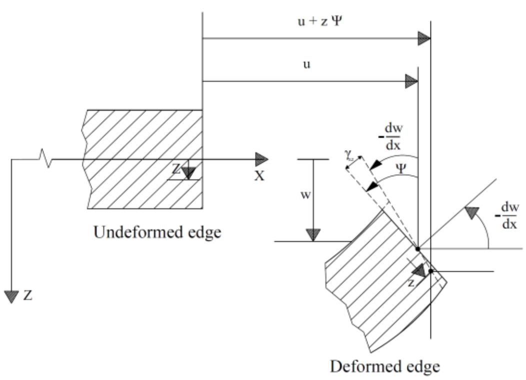

Fig.2: The coordinate system and notation for the Timoshenko beam

plane along the two coordinate directions. According to the Timoshenko beam theory, the governing differential equation for the considered beam can be given by:

$$\frac{d^2}{dx^2}\left(EI\frac{d\psi^T}{dx}\right)=q \tag{16}$$

$$\frac{dw^T}{dx}=\psi-\frac{1}{\kappa AG}\frac{d}{dx}\left(EI\frac{d\psi^T}{dx}\right) \tag{17}$$

where w^T is the transverse displacement of the mid-surface of the Timoshenko beam in the z direction. In addition, EI, A, G and κ are, the flexural rigidity of the beam, the cross-sectional area of the beam, shear modulus, and the sectional shear coefficient, respectively. The superscript T denotes the Timoshenko beam quantities. Based on Hooke's law and the Timoshenko's assumptions, the bending moment–displacement and the shear force–displacement relations are given by:

$$M^T=-EI\frac{d\psi}{dx} \tag{18}$$

$$V^T=-\kappa AG\left(\psi-\frac{dw^T}{dx}\right) \tag{19}$$

Therefore, the Timoshenko beam bending solutions can be obtained by solving the foregoing governing equation together with the natural boundary conditions.

4. Bending Relationships between Euler-Bernoulli and Timoshenko Beams

In this paper, a uniform beam is assumed which is partially restrained against translation and rotation at its ends. The translational restraint is characterized by the spring constant K_{TL}, at one end and K_{TR} at the other end. The rotational restraint is characterized by the spring constants K_{RL} at one end and K_{TR} at the other end, as shown in Fig. 3. Based on the concept of load equivalence for both kinds of beams, it can be deduced that the following equilibrium equations are as follows:

$$\frac{dM}{dx}=V \tag{20}$$

$$\frac{dV}{dx}=-q \tag{21}$$

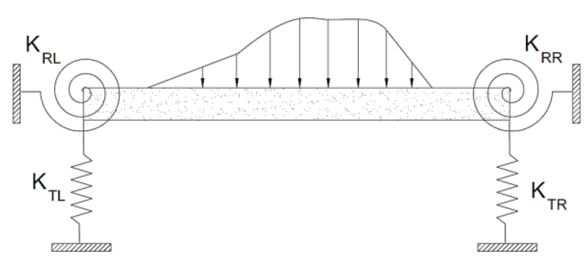

Fig. 3: The beam with general boundary conditions

Substituting Eqs. (9) and (20) into Eq. (21), the governing differential equation for the Euler-Bernoulli beam can be given as follows [18]:

$$-EI\frac{d^4w^E}{dx^4}=\frac{d^2w^E}{dx^2}=-q \tag{22}$$

In addition, by substituting Eqs. (18) and (19) into Eqs. (20) and (21), the governing differential equation for the Timoshenko beam can be given by:

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$$\kappa AG\left(\psi-\frac{dw^T}{dx}\right)=EI\frac{d^2\psi}{dx^2} \tag{23}$$

$$\kappa AG\left(\frac{d\psi}{dx}-\frac{d^2w^T}{dx^2}\right)=q \tag{24}$$

By differentiating Eq. (23) with respect to x and using Eqs. (24) and (18), the following result is obtained:

$$-EI\frac{d^3\psi}{dx^3}=\frac{d^2M^T}{dx^2}=-q \tag{25}$$

The general solution of the differential Eq. (25) considering Eq. (22) can be stated as:

$$\psi=\frac{dw^E}{dx}+C_1\frac{x^2}{2}+C_2x+C_3 \tag{26}$$

On the other hands, it is clear:

$$M^T=M^E-EI\left(C_1x+C_2\right) \tag{27}$$

$$V^T=V^E-EI\,C_1 \tag{28}$$

$$w^T=w^E+\frac{M^E}{\kappa AG}+C_1\left(\frac{x^2}{6}-\frac{EI}{\kappa AG}\right)x+C_2\frac{x^2}{2}+C_3x+C_4 \tag{29}$$

In this paper, the general boundary conditions associated with the beam theory are given below (Ghannadiasl and Mofid, 2015) [22]:

$$\mathrm{V}(0)=-\mathrm{K}_{\mathrm{TL}}w(0) \tag{30}$$

$$\mathrm{M}(0)=\mathrm{K}_{\mathrm{RL}}\theta(0) \tag{31}$$

$$\mathrm{V}(\mathrm{L})=\mathrm{K}_{\mathrm{TR}}w(\mathrm{L}) \tag{32}$$

$$\mathrm{M}(\mathrm{L})=-\mathrm{K}_{\mathrm{RR}}\theta(\mathrm{L}) \tag{33}$$

Furthermore, the bending solution for the Timoshenko beam that is obtained by the above procedure has a general form. By moving the spring constants of the rotational and translational restraint to extreme values (zero and/or infinity), a suitable function can be attained for the desired combinations of end boundary conditions (i.e. simply supported, clamped and free boundary conditions). For example, the displacement of the uniform Timoshenko beam under the uniform distributed load which is partially restrained against translation and rotation at its ends ($K_{RR}=K_{RL}=K_R$ and $K_{TR}=K_{TL}=K_T$) is given below:

$$w^T=-\frac{\mathrm{Lq_0}}{2\mathrm{K_T}}+\frac{\mathrm{Lq_0}}{2\mathrm{K_T}}\mathrm{x}\left(1-\frac{\mathrm{x}}{\mathrm{L}}\right)\left(\frac{12}{\kappa\mathrm{AG}}+\frac{\mathrm{x}(\mathrm{L-x})}{\mathrm{EI}}+\frac{2\mathrm{L}^2}{2\mathrm{EI}+\mathrm{LK_R}}\right) \tag{34}$$

The Timoshenko beam theory is equivalent to the Euler-Bernoulli theory when $\frac{\mathrm{EI}}{\kappa\mathrm{AG}}<<1$. The displacement function for a uniform Euler-Bernoulli and Timoshenko beams with the classical end conditions are listed in Table 1. The displacement function for a uniform Euler-Bernoulli and Timoshenko beams with restrained against translation and rotation at its ends are cited in the Appendix.

5. Numerical Results

In this section, the results of different examples are presented to illustrate the accuracy of the presented theories. The uniform beam with four different boundary conditions at its ends, i.e. simply supported, free, sliding and clamped, are considered. The uniform beam is supposed with the following characteristics:

$$L=6\,m \qquad b=0.35m$$

$$h=\left(1,\frac{1}{2},\frac{1}{3},\frac{1}{5},\frac{1}{8},\frac{1}{10},\frac{1}{20},\frac{1}{30}\right)L$$

$$\nu=0.2 \qquad \kappa=1.224$$

$$q=50000\ kN/m \qquad E=2\times10^{11}\,Pa$$

where q is the external uniform distributed load on the beam. In addition, L, b, h, ν, κ and E are, the length of the beam, the width of the beam, the total depth of the beam, the Poisson's ratio, the sectional shear coefficient, and Young's modulus of elasticity, respectively. Figs. 4-8 compare the values of the deflection of the beam using the Euler–Bernoulli and Timoshenko theories along with the Finite Element (FE) modeling that is carried out using the ABAQUS software package [23]. It can be observed that the values obtained using the Timoshenko beam theory are in good agreement with the Finite Element modeling for all values of the span-to-depth ratio (L/h). Due to ignoring the shear deformation effect, Euler–Bernoulli theory underestimates the deflection of the moderately deep beams (L/h=5). To clearly demonstrate the span-to-depth ratio effectiveness in the accuracy of the theories, a theory performance index is defined as:

$$R_T^E=\left|\frac{\left\langle w_{x_0}\right\rangle_{\text{Timoshenko case}}-\left\langle w_{x_0}\right\rangle_{\text{Euler-Bernoullicase}}}{\left\langle w_{x_0}\right\rangle_{\text{Timoshenkocase}}}\right|\times100 \tag{35}$$

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$$R_E^{FE} = \left| \frac{\langle w_{x_0} \rangle_{\text{Finite Element case}} - \langle w_{x_0} \rangle_{\text{Euler-Bernoulli case}}}{\langle w_{x_0} \rangle_{\text{Finite Elementi case}}} \right| \times 100 \quad (36)$$

$$R_T^{FE} = \left| \frac{\langle w_{x_0} \rangle_{\text{Finite Element case}} - \langle w_{x_0} \rangle_{\text{Timoshenko case}}}{\langle w_{x_0} \rangle_{\text{Finite Element case}}} \right| \times 100 \quad (37)$$

where $\langle w_{x_0} \rangle_{\text{Euler-Bernoulli case}}$ denotes the value of the maximum displacement of the Euler–Bernoulli theory in x_0, $\langle w_{x_0} \rangle_{\text{Timoshenko case}}$ denotes the maximum value of the displacement of the Timoshenko theory in x_0, and $\langle w_{x_0} \rangle_{\text{Finite Element case}}$ presents the maximum value of the displacement of the Finite Element (FE) modelling in x_0. To illustrate the theory performance index on the bending response of beams under uniform load, the theory performance indexes are presented in Table 2. It is observed that the beam theory can be considered as the classical beam theory when the length to thickness ratio are greater than 20. From Table 2, it illustrates that in the Timoshenko beam theory with simply supported, fixed– free, and fixed - sliding boundary conditions, the theory performance index for L/h=3 are 0.79, 1.81 and 2.54, respectively. Table 2 clearly presents that the classical beam theory is almost the same as the Timoshenko beam theory when the thickness to length ratio reaches the limit of 1/20. Also, it can be seen that, by increasing the thickness to length ratio about L/h=2, the R_T^{FE} will reduce the accuracy of the Timoshenko beam theory. On the other hand, the results indicate the significant effect of the boundary conditions, not only on the beam response, but also on the theory performance. Comparing the results of the clamped– simply supported beam and the clamped – clamped beam show that the effect of the support is more significant.

Table. 1: The displacement function for the uniform Euler-Bernoulli and Timoshenko beam with classical end conditions

End boundary conditions	Euler-Bernoulli beam	Timoshenko beam
Pined-pinned	$w^E = \frac{q_0 x}{24EI}\left(L^3 - 2Lx^2 + x^3\right)$	$w^T = \frac{\beta}{\lambda}\left(12EI + \alpha\left(L^2 + Lx - x^2\right)\right)$
Fixed-fixed	$w^E = \frac{q_0 x^2}{24EI}(L - x)^2$	$w^T = \frac{\beta}{\lambda}\left(12EI + \alpha x(L - x)\right)$
Pinned-fixed	$w^E = \frac{q_0 x}{48EI}(L - x)^2(L + 2x)$	$w^T = \frac{\beta}{2\lambda\left(3EI + \alpha L^2\right)}\begin{pmatrix} 72(EI)^2 L^2 \alpha^2 (L - x)(L + 2x) + \\ 6\alpha EI\left(5L^2 + Lx - x^2\right) \end{pmatrix}$
Fixed-free	$w^E = \frac{q_0 x^2}{24EI}\left(6L^2 - 4Lx + x^2\right)$	$w^T = \frac{qx}{\lambda}\left(12EI(2L - x) + \alpha x\left(6L^2 - 4Lx + x^2\right)\right)$
Fixed-sliding	$w^E = \frac{q_0 x^2}{24EI}(-2L + x)^2$	$w^T = \frac{qx}{\lambda}(2L - x)\left(12EI + \alpha x(2L - x)\right)$

$$\alpha = GA\kappa \qquad \beta = qx(L - x) \qquad \lambda = 24EIGA\kappa$$

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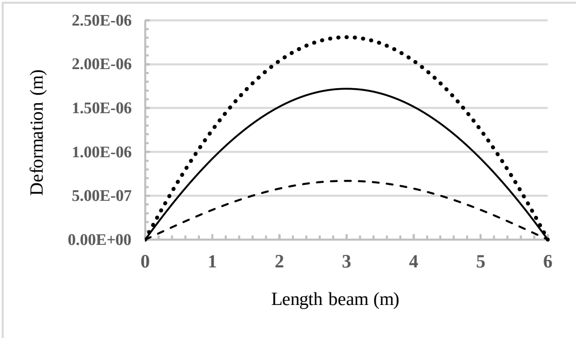
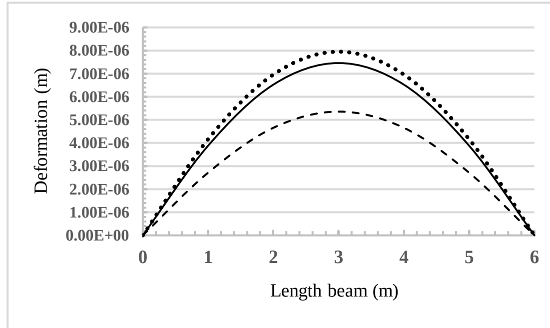
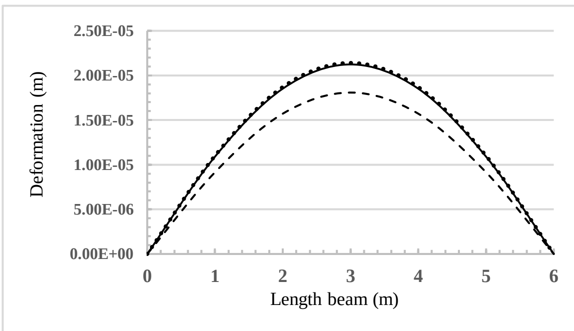


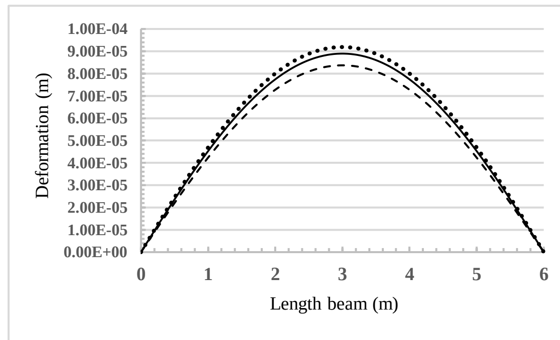
A: L/h=1



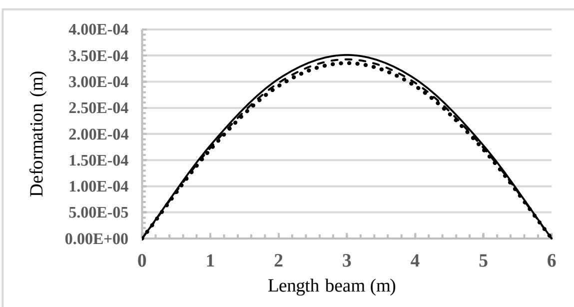
B: L/h=2



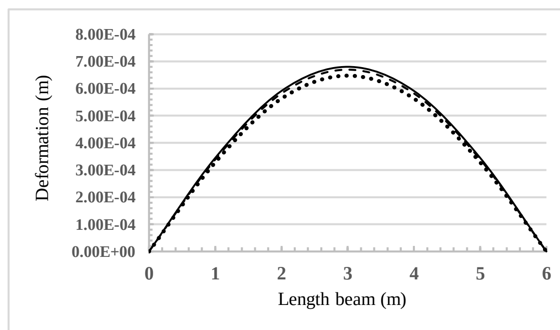
C: L/h=3



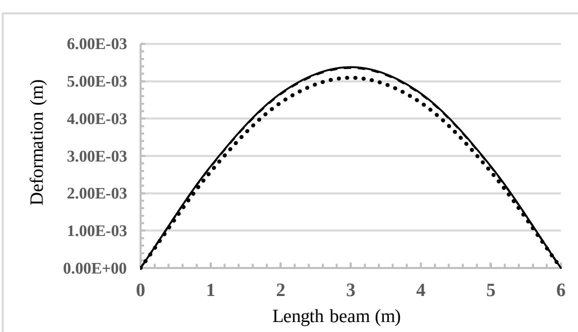
D: L/h=5



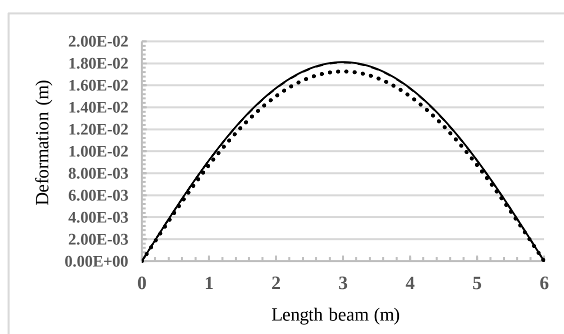
F: L/h=8



G: L/h=10



H: L/h=20



I: L/h=30

Euler-Bernoulli Timoshenko Finite element

Fig. 4: The deflection of the simply supported - simply supported beam under uniform distributed load

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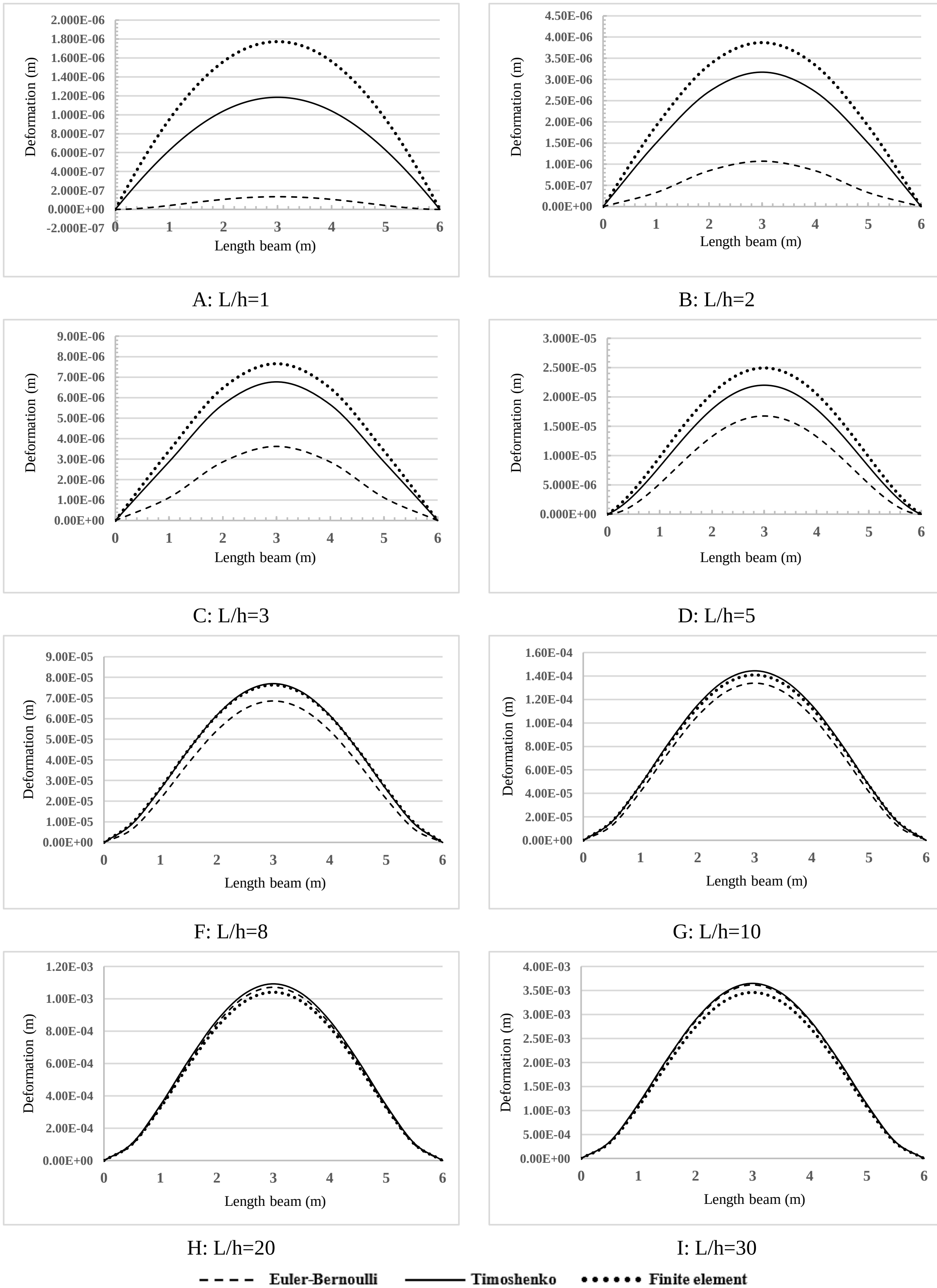

Fig. 5: The deflection of the fixed-fixed beam under uniform distributed load

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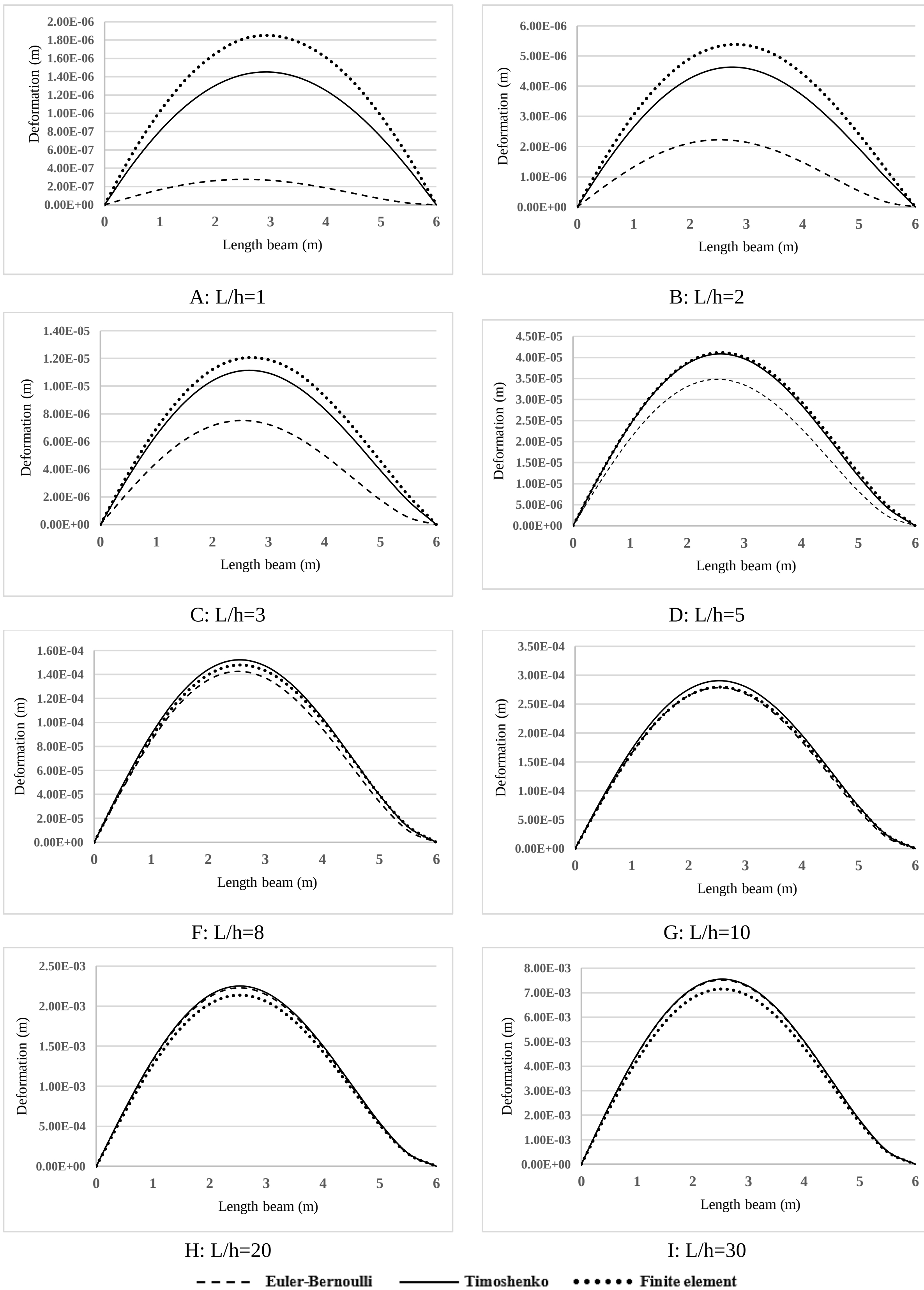

Fig. 6: The deflection of the simply supported-fixed beam under uniform distributed load

[DOI: 10.29252/nmce.2.2.63]

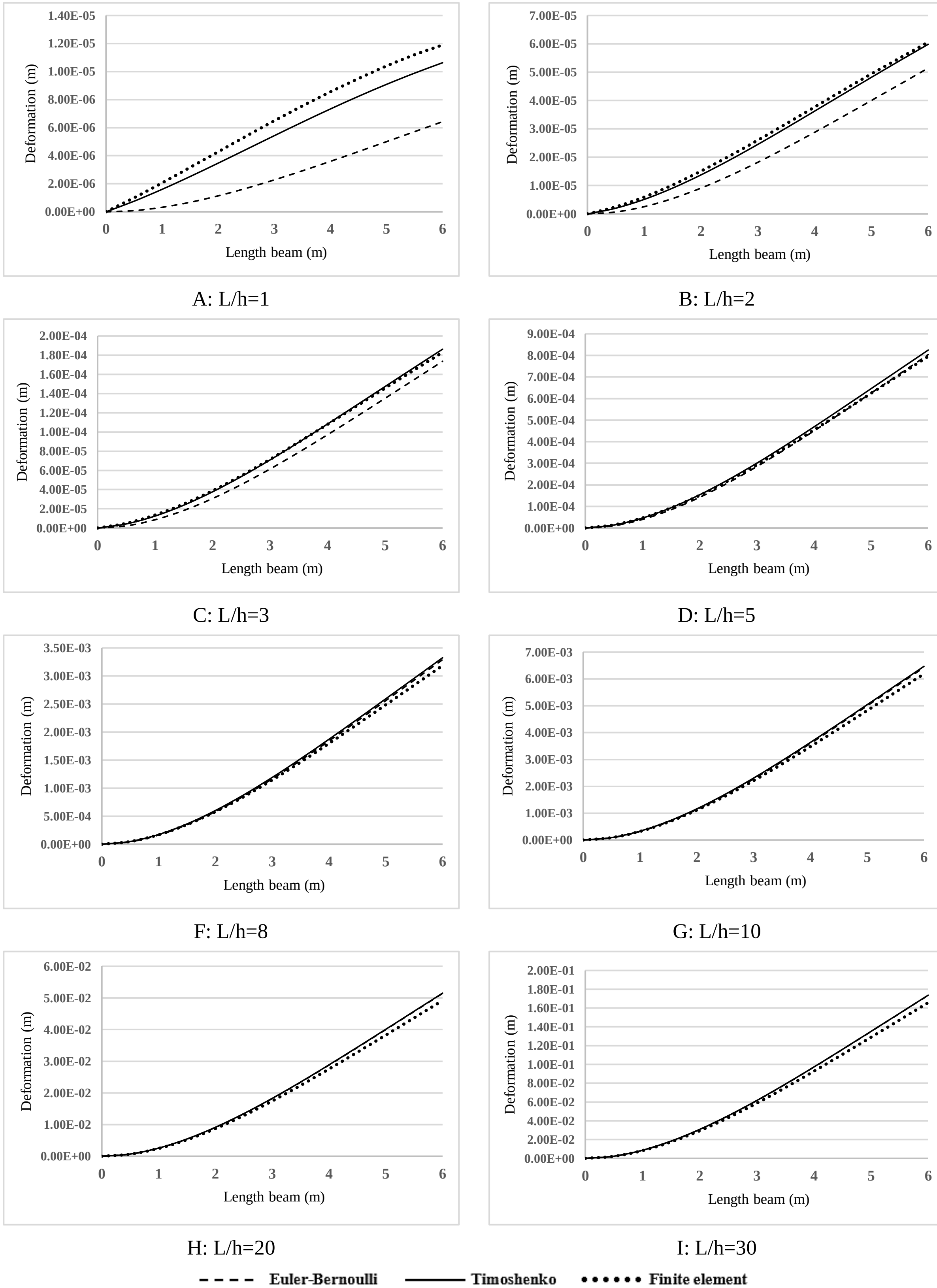

Fig. 7: The deflection of the fixed-free beam under uniform distributed load

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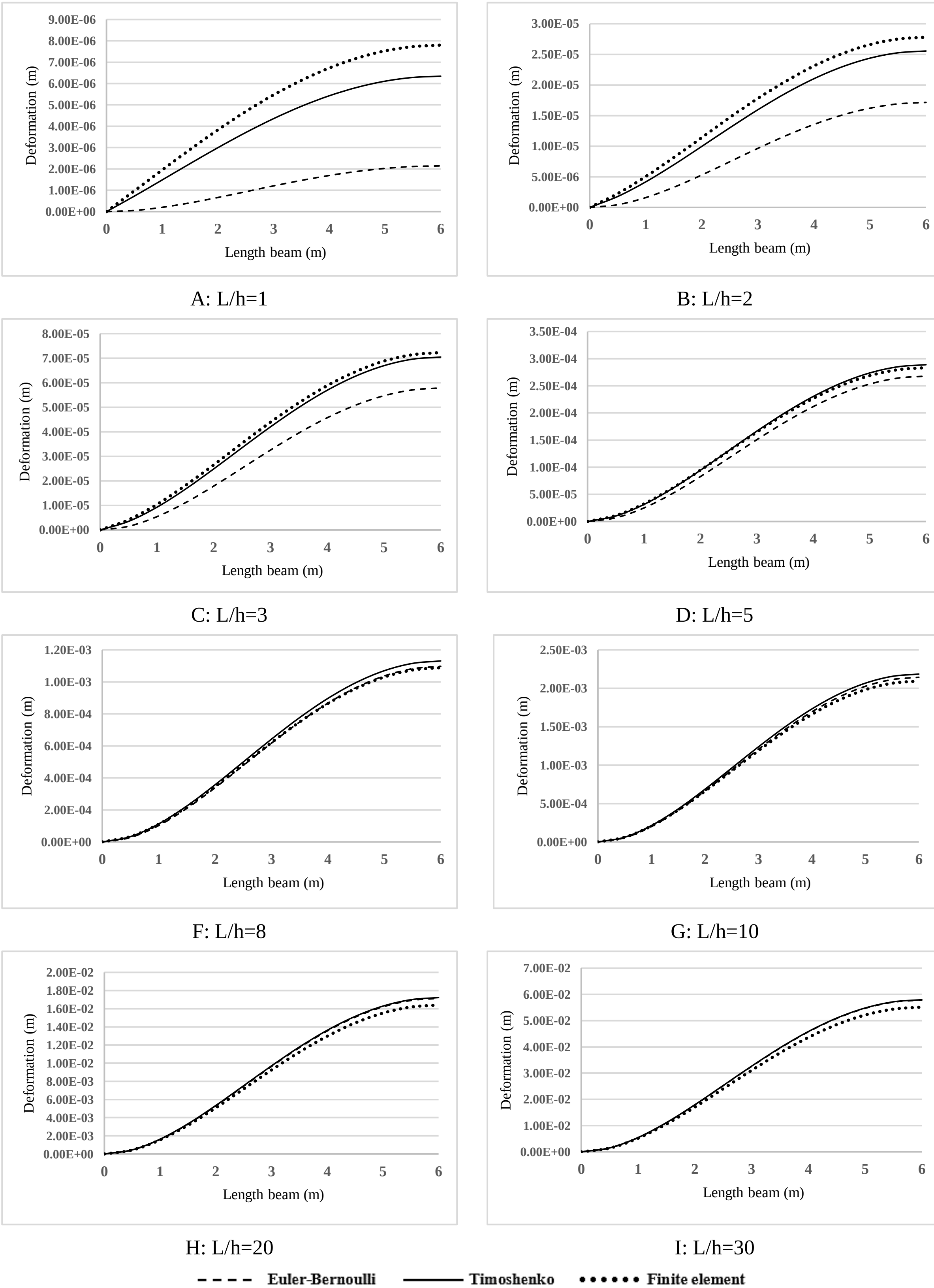

Fig. 8: The deflection of the fixed- sliding beam under uniform distributed load

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Table 2: The theory performance index on the bending response of beams with classical end conditions under uniform load

Boundary conditions	Theory performance index	**L/h=1**	**L/h=2**	**L/h=3**	**L/h=5**	**L/h=8**	**L/h=10**	**L/h=20**	**L/h=30**
simply supported - simply supported	R_T^E	61.07	28.13	14.84	5.90	2.39	1.54	0.39	0.17
	R_E^{FE}	70.99	32.58	15.51	8.91	2.13	3.41	5.16	4.74
	R_T^{FE}	25.48	6.19	0.79	3.19	4.63	5.03	5.57	4.93
fixed fixed	R_T^E	88.69	66.23	46.57	23.88	10.92	7.27	1.92	0.86
	R_E^{FE}	92.44	72.31	52.75	32.86	10.13	4.96	2.94	4.54
	R_T^{FE}	33.19	18.03	11.57	11.79	0.89	2.50	4.95	5.45
simply supported -fixed	R_T^E	81.53	53.36	33.92	15.66	6.78	4.45	1.15	0.51
	R_E^{FE}	85.52	60.02	39.23	16.50	4.21	0.81	4.12	5.09
	R_T^{FE}	21.60	14.28	8.04	1.00	2.75	3.80	5.33	5.64
fixed-free	R_T^E	39.53	14.04	6.77	2.55	1.01	0.65	0.16	0.07
	R_E^{FE}	45.98	15.13	5.08	1.03	3.31	3.85	4.58	4.72
	R_T^{FE}	10.67	1.27	1.81	3.67	4.37	4.53	4.75	4.80
fixed - sliding	R_T^E	66.23	32.89	17.89	7.27	2.97	1.92	0.49	0.22
	R_E^{FE}	72.53	38.33	19.98	5.48	0.76	2.32	4.48	4.89
	R_T^{FE}	18.66	8.11	2.54	1.93	3.85	4.33	4.99	5.11

Also, the influence of the spring support on the theory performance index (R_T^E) is evaluated. For this purpose, the beam is assumed with spring supports, K_T and K_R. The stiffness of the spring is taken as having the same values at both of the supports of the beam. The beam characteristics are as follows:

$$L = 6\ m \qquad b = 0.35\,m$$
$$K_{TL} = K_{TR} = K$$
$$K_{RL} = K_{RR} = K$$
$$h = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{8}, \frac{1}{10}, \frac{1}{15}, \frac{1}{20}, \frac{1}{25}, \frac{1}{30}\right) L$$
$$\nu = 0.2 \qquad \kappa = 1.224$$
$$q = 50000\ kN/m \qquad E = 2\times 10^{11}\,Pa$$

To illustrate the theory performance index on the bending response of beam with spring supports under uniform load, the theory performance index is presented in Table 3.

It is observed that the beam with spring supports can be considered as clamped at both ends when the values of K_T/EI and K_R/EI are greater than 1000. From Table 3, it illustrates that in the beam with $\frac{K}{EI} = 50$ and $\frac{K}{EI} = 500$, the theory performance indices are 10.83 and 10.91 for L/h=8, respectively. Table 3 clearly presents that the values of the theory performance indices are almost the same when the rotational springs' stiffness is larger than 500EI.

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Table 3: The theory performance index (R_T^E) on the bending response of beams with spring supports under uniform load

	L/h=1	L/h=2	L/h=3	L/h=5	L/h=8	L/h=10	L/h=15	L/h=20	L/h=25	L/h=30
5EI	87.97	64.65	44.84	22.64	10.26	6.82	3.15	1.80	1.16	0.81
10EI	88.29	65.34	45.59	23.17	10.54	7.01	3.24	1.85	1.19	0.83
25EI	88.52	65.85	46.14	23.57	10.75	7.16	3.31	1.89	1.22	0.85
50EI	88.60	66.03	46.35	23.72	10.83	7.21	3.34	1.91	1.23	0.86
100EI	88.65	66.13	46.46	23.80	10.87	7.24	3.35	1.91	1.23	0.86
500EI	88.68	66.21	46.54	23.86	10.91	7.27	3.37	1.92	1.24	0.86
1000EI	88.69	66.22	46.55	23.87	10.91	7.27	3.37	1.92	1.24	0.86
5000EI	88.69	66.22	46.56	23.88	10.92	7.27	3.37	1.92	1.24	0.86
10000EI	88.69	66.22	46.56	23.88	10.92	7.27	3.37	1.92	1.24	0.86
∞	88.69	66.23	46.57	23.88	10.92	7.27	3.37	1.92	1.24	0.86

6. Conclusion

In this paper, the bending solution of the thin and thick beams elastically restrained against rotation and translation using various theories is presented. An accurate and direct modeling technique is introduced for modeling beam with arbitrary boundary conditions. Some numerical examples are shown in order to present the efficiency and simplicity of the various theories. Also, the results of the various beam theories are compared with the results of the finite element method (ABAQUS). Based on the results, the classical beam theory is almost the same as the Timoshenko beam theory when the thickness to length ratio reaches the limit of 1/20. Also, it can be seen that by increasing the thickness to length ratio about L/h=2, the R_T^{FE} will reduce the accuracy of the Timoshenko beam theory. On the other hand, the results indicate the significant effect of the boundary conditions, not only on the beam response, but also on the theory performance. Comparing the results of the clamped– simply supported beam and the clamped – clamped beam show that the effect of the support is more significant.

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Appendix:

A. Euler-Bernoulli beam with restrained against translation and rotation at its ends:

$$w^E = 2EIq\,\frac{\left(\eta A_1 + K_{LR}A_5 + \delta\left(A_6 + 2\eta(L+x)A_7\right)K_{RT}\right)}{\left(4\eta\left(2\eta K_{LT}A_8 + \left(-4EIA_9 + A_{10}\right)K_{RT}\right)\right)}$$

B. Timoshenko beam with restrained against translation and rotation at its ends:

$$w^T = -2EIq\,\frac{\left(\eta\alpha\left(4\eta\alpha LK_{RR} + A_{11}\right) + A_{22} + 2\eta\alpha\left(A_{20}A_{21}\right)\right)}{4\eta\alpha\left(A_{23} + \left(A_{24}\right) + K_{LR}A_{25}\right)K_{RT}}$$

where

$$\alpha = GA\kappa \qquad \beta = qx(L-x)$$

$$\lambda = 24EIGA\kappa \qquad \eta = 6EI \qquad \delta = (L-x)$$

$$A_1 = \left(-4\eta LK_{RR} + xK_{LT}A_2 - \delta\left(A_3 + A_4\right)K_{RT}\right)$$

$$A_2 = \left(2\eta L^2 + \left(8L^3 - 4Lx^2 + x^3\right)K_{RR}\right)$$

$$A_3 = \eta L^2\left(-2\eta + x\left(L^2 + Lx - x^2\right)K_{LT}\right)$$

$$A_4 = \delta\left(-\eta\left(3L^2 + 2Lx + x^2\right) + L^3x(L+2x)K_{LT}\right)K_{RR}$$

$$A_5 = \left(2\eta\left(-4\eta\mathrm{L}\left(\mathrm{EI} + \mathrm{LK}_{\mathrm{RR}}\right)x^2K_{LT}\left(EI\left(6L^2 - 4Lx + x^2\right) + L(-2L+x)^2K_{RR}\right)\right)\right)$$

$$A_6 = L^3x^2K_{LT}\left(-\eta L + 4EIx + L(-L+x)K_{RR}\right)$$

$$A_7 = \left(EI\left(5L^2 - x^2\right) + L\delta(L+x)K_{RR}\right)$$

$$A_8 = \left(EIK_{RR} + K_{LR}\left(EI + LK_{RR}\right)\right)$$

$$A_9 = \left(-3EIK_{RR} + L^2K_{LT}\left(3EI + LK_{RR}\right)\right)$$

$$A_{10} = K_{LR}\left(2\eta\left(EI + LK_{RR}\right) - L^3K_{LT}\left(4EI + LK_{RR}\right)\right)$$

$$A_{11} = xK_{LT}\left(-2\eta\alpha L^2 - (-2L+x)A_{12}K_{RR}\right)$$

$$A_{12} = \left(-2\eta + \alpha\left(-4L^2 - 2Lx + x^2\right)\right)$$

$$A_{13} = \left(-2\eta\alpha + x\left(2\eta + \alpha\left(L^2 + Lx - x^2\right)\right)K_{LT}\right)$$

$$A_{14} = \left(\left(\eta\alpha\left(-2\eta(L+x) + \alpha\left(-3L^3 + L^2x + x^2L + x^3\right)\right) + A_{15}\right)K_{RR}\right)K_{RT}$$

$$A_{15} = Lx\left(72(EI)^2\alpha^2L^2(L-x)(L+2x) + \eta\alpha\left(5L^2 + Lx - x^2\right)\right)K_{LT}$$

$$A_{16} = \left(2\eta(2L-x) + \alpha x\left(6L^2 - 4Lx + x^2\right)\right)$$

$$A_{17} = 2\eta\alpha L(2L-x) + \left(2\eta + \alpha(2L-x)x\right)$$

$$A_{18} = 2EI\left(72(EI)^2 + \alpha^2(3L-2x)x + \eta\alpha\left(5L^2 + Lx - x^2\right)\right)$$

$$A_{19} = L\left(2\eta + \alpha L^2\right)\left(2\eta + \alpha(L-x)x\right)$$

$$A_{20} = 4\eta\alpha L\left(EI + LK_{RR}\right) - (L-x)(L+x)$$

$$A_{21} = \left(EI\left(2\eta + \alpha\left(5L^2 - x^2\right)\right) + L\left(2\eta + \alpha(L-x)(L+x)\right)K_{RR}\right)K_{RT}$$

$$A_{22} = (\delta)\left(\eta\alpha L^2\left(A_{13}\right)\right) + \left(A_{14}\right) + K_{LR}xK_{LT}\left(-12(EI)^2\alpha\left(A_{16} - A_{17}K_{RR} + L(L-x)\left(A_{18} + A_{19}\right)K_{RR}\right)K_{RT}\right)$$

$$A_{23} = 2\eta\alpha K_{LT}\left(EIK_{RR} + K_{LR}\left(EI + LK_{RR}\right)\right)$$

$$A_{24} = 4EI\left(3EI\alpha K_{RR} - LK_{LT}\left(3EI\alpha L + \left(3EI + \alpha L^2\right)K_{RR}\right)\right)$$

$$A_{25} = \left(2\eta\alpha\left(EI + LK_{RR}\right) - LK_{LT}\left(4EI\left(3EI + \alpha L^2\right) + L\left(2\eta + \alpha L^2\right)K_{RR}\right)\right)$$

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