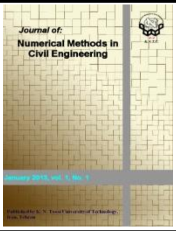

Numerical Methods in Civil Engineering

Stability and vibration analyses of tapered columns resting on one or two-parameter elastic foundations

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ARTICLE INFO

Article history:
Received:
July 2014.
Revised:
October 2014.
Accepted:
November 2014.

Keywords:
Stiffness matrix,
Power series solution,
Non-prismatic column,
Buckling, vibration.

Abstract:

This paper presents a generalized numerical method to evaluate element stiffness matrices needed for the free vibration and stability analyses of non-prismatic columns resting on one- or two-parameter elastic foundations and subjected to variable axial load. For this purpose, power series approximation is used to solve the fourth–order differential equation of non-prismatic columns with variable geometric parameters. Then, the shape functions are obtained exactly by deriving the deformation shape of the column as power series form. Finally, the element stiffness matrices are determined by means of the principle of virtual work along the columns axis. In order to demonstrate the accuracy and the efficiency of presented method, several numerical examples including in the free-vibration and buckling analysis of non-prismatic columns, portal frame, and gable frame are presented and obtained results compared with the results of other available numerical and theoretical approaches. The method can be applied for the buckling load and natural frequencies computation of uniform members as well as non- prismatic members.

1. Introduction

Members with variable cross section are widely used as columns in many engineering structures such as high-rise building, aeronautical structures, cranes and other application fields. In order to increase buckling strength and reduce vibration effects, appropriate distribution of weight has to be found in order to utilize the structural material more efficiently. Vibration and stability analysis of non-prismatic columns has been studied by several researchers because of its relevance to aeronautical, civil and architectural engineering. Closed form solutions of the fourth-order differential equation governing the stability or vibration behaviour of tapered beams are often difficult, and exist only for limited cases. They are available and extensively discussed in reference books [1-3]. Many numerical techniques such as finite element method, finite difference method and power series approach are used to solve the vibration and stability of these structures .Some of them are based on exact formulations [4-7]. The dynamic stiffness matrices for non-uniform cross-section beams were derived by Banerjee and Williams [4] using Bessel's functions. By means of the flexibility–stiffness transformation approach, the exact stiffness matrix of a non-uniform beam was derived by Frieman and Kosmatka [5]. The power series method was adopted by Al-Sadder [6] in deriving the exact stability functions of beam-column tapered elements.

A finite element model is proposed in Karabalis [7] for dynamic and stability analysis of plane tapered beams with variable depth. It is on an exact flexural and axial stiffness matrix. Some other models are investigated for the stability and the vibration of beams with variable cross sections [8-10]. These models are based on approximate formulations. These models are a good compromise between the analytical procedures and numerical methods and are efficient tools in design. Based on the energy method, the modified vibration mode shapes were used by Rahai [8] in buckling analysis. Jategaonkar [9] adopted the Galerkin's method in vibration behaviour of a beam with varying section properties. Saffari [10] adopted the effective length

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method in the stability of plane and gabled frames with

The study of stability and vibration analyses of beam-columns resting on elastic foundations is essential in many problems related to soil-structure interaction (the foundation of buildings, pipelines embedded in soil, highway pavements, etc.). The Winkler hypothesis is the most used mechanical model in the solution of these problems. In this model, the elastic foundation is considered as limiting case of an infinitely dense distribution of linear springs. Closed form solutions of uniform beam-columns resting on this type of elastic foundation were studied in reference books [2-3]. A modified Valsov model was applied to free vibration analysis of beam resting on elastic foundation by Ayvaz [11]. However, the modeling of the soil using Winkler's theory was considered inadequate in several problems as this model overlooks the soil cohesion. In order to improve this weakness, various two-parameter elastic foundation models were developed. In these models, interactions between springs were considered. For buckling or free-vibration analysis of beams on elastic foundations, Eisenberger [12] and Matsunaga [13] proposed a method based on power series expansion of displacement components. Static or dynamic stiffness matrices of non-uniform members resting on variable elastic foundations were derived by Girgin [14]. This method was performed by means of a generalized numerical method which is based on the well-known Mohr method. The stiffness matrices for beams on three parameter elastic foundation are developed on the basis of exact solution of the governing differential equation by Avramidis [15].

The stability of thin-walled tapered beams with arbitrary cross sections by means of the power series method was investigated in Asgarian [16]. The previous method has been extended to stability and free vibration behaviour for beams with non-symmetric cross-sections and arbitrary boundary conditions in Asgarian [17]. In these studies only rigid supports are considered.

The aim of this study is to determine the stiffness matrices for the critical buckling loads and free vibration analysis of non-prismatic column members resting on one- or two-parameter elastic foundations and subjected to variable axial forces based on power series expansions. The important points presented are summarized as follows:

1. The power series expansions are used to solve the fourth-order equilibrium differential equation of non-prismatic column members with variable geometric parameters. In this regard, it is assumed that the functions which describe the beam's variable parameters such as: flexural rigidity, mass and loads can be expanded into power series form. Explicit expressions for deformation shape components are determined based on aforementioned method.

variable cross sections.

2. The terms of stiffness and mass matrices can be determined by means of the shape functions resulting from its nodal displacements and the principle of virtual work along the column axis.

Several numerical examples are presented in order to measure the accuracy and verify the validity of proposed method, and the results compared with the results of other available investigations. The main advantages of this method are that the proposed method can be applied in various form of non-prismatic member. Moreover, this method does not require any complex and time consuming analysis.

2. Derivation and Formulation of Basic Equations

In the study, a non-prismatic column element of length $\boldsymbol{L}$ with variable bending rigidity $\boldsymbol{EI(x)}$ subjected to static axial load $N(x)$ is considered (Fig. 1). The beam stays on two-parameter elastic foundations. The first elastic series is translational with variable elastic constant $\boldsymbol{K(x)}$. The second is rotational with related stiffness $\boldsymbol{K_1}(x)$. In plane bending, the element has two degrees of freedom at each node: vertical translations in the nodal y direction and rotation about z-axis (the two nodes by which the finite element can be assembled into structure are located at its ends) (Fig.2). The displacement of the column element is related to its four Degrees Of Freedom (DOF) by:

$$y(x,t)=\sum_{j=1}^{4} y_j(t)\psi_j(x) \tag{1}$$

The terms of the element mass and stiffness matrices can be found from the derivatives of the interpolation functions. Where the function $\psi_j(x)$ defines the displacement of the element from applying unit translation or rotation, at each of the four degrees of freedom, while constraining the other three nodal displacements are shown in Fig.3a-d. Thus $\psi_j(x)$ satisfies the following boundary conditions as follows:

$$j=1 \qquad \psi_1(0)=1; \quad \psi_1'(0)=\psi_1(L)=\psi_1'(L)=0 \tag{2-a}$$
$$j=2 \qquad \psi_2'(0)=1; \quad \psi_2(0)=\psi_2(L)=\psi_2'(L)=0 \tag{2-b}$$
$$j=3 \qquad \psi_3(L)=1; \quad \psi_3'(0)=\psi_3(0)=\psi_3'(L)=0 \tag{2-c}$$
$$j=4 \qquad \psi_4'(L)=1; \quad \psi_4'(0)=\psi_4(0)=\psi_4'(L)=0 \tag{2-d}$$

These shape functions could be taken any arbitrary shapes which satisfy the boundary conditions of element and internal continuity requirements. With these four interpolation functions, the exact deflected shape of the

[DOR: 20.1001.1.23454296.2016.1.2.5.9]
[DOI: 10.29252/nmce.1.2.57]

column element can be expressed in terms of its nodal displacements. They can be obtained for a non-prismatic column as illustrated below. Neglecting shear deformation, the differential flexural equilibrium equation with variable coefficients for a column resting on two parameter elastic foundations and loaded by variable axial force is:

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2y}{dx^2}\right]+\frac{d}{dx}\left[(N(x)-K_1(x))\frac{dy}{dx}\right] +K(x)y(x)=0 \tag{3}$$

In the last equation, due to presence of non-prismatic elements, all variable properties including the moment of inertia of column's cross -section *I(x)* , the axial load *N(x)* , the function of second parameter of two-parameter elastic foundation $K_1(x)$, Winkler type foundation modulus *K(x)* can be presented in power series form as follows :

$$I(x)=\sum_{i=0}^{\infty} I_i x^i \qquad ,K(x)=\sum_{i=0}^{\infty} K_i x^i$$
$$K_1(x)=\sum_{i=0}^{\infty} K_{1i} x^i \qquad ,N(x)=\sum_{i=0}^{\infty} N_i x^i \tag{4}$$

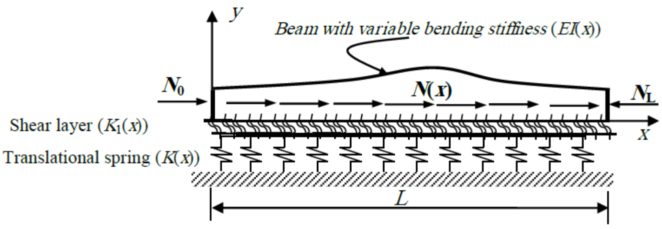


Fig. 1: Non-prismatic column on a two-parameter elastic foundation, subjected to variable axial force.

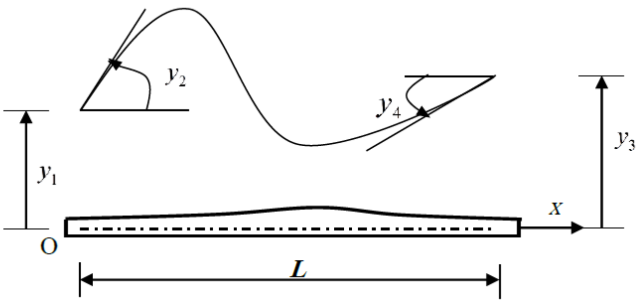


Fig. 2: Degrees of freedom for a column element in global coordinate system.

Introducing a new non-dimensional variable $\xi=\frac{x}{L}$, Eq. (4) can be written as:

$$I(\xi)=\sum_{i=0}^{\infty} I_i L^i \xi^i \qquad ,K(\xi)=\sum_{i=0}^{\infty} K_i L^i \xi^i$$
$$N(\xi)=\sum_{i=0}^{\infty} N_i L^i \xi^i \qquad ,K_1(\xi)=\sum_{i=0}^{\infty} K_{1i} L^i \xi^i \tag{5}$$

Substituting Eq. (4) and (5) into Eq. (3) yields to following differential equilibrium equation:

$$\frac{d^2}{d\xi^2}\left[E\left(\sum_{i=0}^{\infty} I_i L^i \xi^i\right)\frac{d^2y}{d\xi^2}\right]$$
$$+L^2\frac{d}{d\xi}\left[\left(\sum_{i=0}^{\infty} N_i L^i \xi^i-\sum_{i=0}^{\infty} K_{1i} L^i \xi^i\right)\frac{dy}{d\xi}\right] \tag{6}$$
$$+\left(L^4\right)\left[\left(\sum_{i=0}^{\infty} K_i L^i \xi^i\right)y(\xi)\right]=0$$

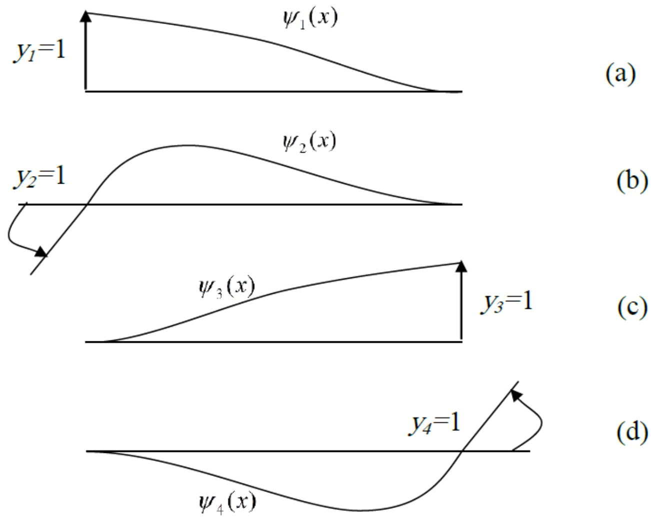


Fig. 3: Interpolation functions.

The general solution of (Eq. (6)) can be presented in the following form of power series in terms of the variable $\xi=0$:

$$y(\xi)=\sum_{j=0}^{\infty} b_j \xi^i \tag{7}$$

Then;

$$\frac{dy(\xi)}{d\xi}=\sum_{j=1}^{\infty} j b_i \xi^{j-1}=\sum_{j=0}^{\infty}(j+1)b_{j+1}\xi^j \tag{8}$$

$$\frac{d^2y(\xi)}{d\xi^2}=\sum_{j=2}^{\infty} j(j-1)b_j\xi^{j-2}=\sum_{j=0}^{\infty}(j+1)(j+2)b_{j+2}\xi^j \tag{9}$$

Furthermore, introducing new variables:

$$I_i^* = I_i L^i \quad ,N_i^* = N_i L^i \quad ,K_i^* = K_i L^i \quad ,K_{1i}^* = K_{1i} L^i \quad (10)$$

And substituting Eq. (7)-(10) into Eq. (6) the following equation can be found:

$$\begin{aligned}&\frac{d^2}{d\xi^2}\left[E\left\{\sum_{i=0}^{\infty} I_i^*\xi^i\right\}\left\{\sum_{j=0}^{\infty} b_{j+2}(j+2)(j+1)\xi^j\right\}\right]\\&+L^2\frac{d}{d\xi}\left[\left\{\sum_{i=0}^{\infty} N_i^*\xi^i-\sum_{i=0}^{\infty} K_{1i}^*\xi^i\right\}\left\{\sum_{j=0}^{\infty} b_{j+1}(j+1)\xi^j\right\}\right]\\&+\left(L^4\right)\left[\left\{\sum_{i=0}^{\infty} K_i^*\xi^i\right\}\left\{\sum_{j=0}^{\infty} b_j\xi^j\right\}\right]=0\end{aligned} \tag{11}$$

By multiplying of the two series in each terms of equation (11), the following expression can be obtained:

$$\begin{aligned}\sum_{j=0}^{\infty}\Bigg[&E\sum_{i=0}^{j+2} I_i^*(j-i+4)(j-i+3)(j+2)(j+1)b_{j-i+4}\\&+L^2\sum_{i=0}^{j+1}(N_i^*-K_{1i}^*)(j-i+2)(j+1)b_{j-i+2}\\&+L^4\sum_{i=0}^{j} K_i^* b_{j-i}\Bigg]\xi^j=0\end{aligned} \tag{12}$$

Or

$$\begin{aligned}&\sum_{j=0}^{\infty}\Big[EI_0^*(j+4)(j+3)(j+2)(j+1)b_{j+4}\\&+E\sum_{i=1}^{j+2} I_i^*(j-i+4)(j-i+3)(j+2)(j+1)b_{j-i+4}\\&+L^2\sum_{i=0}^{j+1}(N_i^*-K_{1i}^*)(j-i+2)(j+1)b_{j-i+2}\\&+L^4\sum_{i=0}^{j} K_i^* b_{j-i}\Big]\xi^j=0\end{aligned} \tag{13}$$

To satisfy last expression for every value of ξ, one must have the following recurrence formula about the term b_{j+4} :

$$\begin{aligned}b_{j+4}&=\frac{-1}{EI_0^*(j+4)(j+3)(j+2)(j+1)}\\&\times\Bigg[E\sum_{i=1}^{j+2} I_i^*(j-i+4)(j-i+3)(j+2)(j+1)b_{j-i+4}\\&+L^2\sum_{i=0}^{j+1}(N_i^*-K_{1i}^*)(j-i+2)(j+1)b_{j-i+2}+L^4\sum_{i=0}^{j} K_i^* b_{j-i}\Bigg]\\&\qquad for\ j=0,1,2,........\end{aligned} \tag{14}$$

With the recurrence formula, the solution of Eq. (3) could be obtained explicitly in terms of the four constants (b_0, b_1, b_2, b_3). The general solution of Eq. (3) can be written in the following form:

$$Y(\xi)=b_0 y_0(\xi)+b_1 y_1(\xi)+b_2 y_2(\xi)+b_3 y_3(\xi) \tag{15}$$

In which $y_i\ (i=0,1,2,3)$ are the fundamental solutions of Eq. (13). The first few terms of y_i are listed as examples in Appendix A. Knowing that the first four coefficients (b_0, b_1, b_2, b_3) are functions of the displacements DOF. Thus, the displacement perpendicular to the beam's axis $Y(\xi)$ can be obtained from the right and left ends boundary conditions of element. The equivalent shape functions corresponding to the four sets boundary conditions of Eq. 2 can be derived as follows:

$$\begin{aligned}\psi_1(\xi)&=y_0(\xi)+\left(\frac{(y_0(1)y_3'(1)-y_3(1)y_0'(1))}{(y_3(1)y_2'(1)-y_2(1)y_3'(1))}\right)y_2(\xi)\\&+\left(-\frac{(y_0(1)y_2'(1)-y_2(1)y_0'(1))}{(y_3(1)y_2'(1)-y_2(1)y_3'(1))}\right)y_3(\xi)\end{aligned} \tag{16}$$

$$\begin{aligned}\psi_2(\xi)&=Ly_1(\xi)+\left(\frac{L(y_1(1)y_3'(1)-y_3(1)y_1'(1))}{(y_3(1)y_2'(1)-y_2(1)y_3'(1))}\right)y_2(\xi)\\&+\left(-\frac{L(y_1(1)y_2'(1)-y_2(1)y_1'(1))}{(y_3(1)y_2'(1)-y_2(1)y_3'(1))}\right)y_3(\xi)\end{aligned} \tag{17}$$

$$\begin{aligned}\psi_3(\xi)&=\left(-\frac{y_3'(1)}{(y_3(1)y_2'(1)-y_2(1)y_3'(1))}\right)y_2(\xi)\\&+\left(\frac{y_2'(1)}{(y_3(1)y_2'(1)-y_2(1)y_3'(1))}\right)y_3(\xi)\end{aligned} \tag{18}$$

$$\begin{aligned}\psi_4(\xi)&=\left(\frac{L\,(y_3(1))}{(y_3(1)y_2'(1)-y_2(1)y_3'(1))}\right)y_2(\xi)\\&+\left(-\frac{L\,(y_2'(1))}{(y_3(1)y_2'(1)-y_2(1)y_3'(1))}\right)y_3(\xi)\end{aligned} \tag{19}$$

The illustrated shape functions in Fig. (3 a-d) and the principle of internal virtual work can be used in formulating the element stiffness matrices of non-prismatic members including geometrical and generalized stiffness matrices needed for stability analysis. In the case of the column resting on two parameter elastic foundations, the terms of these matrices are the following:

$$K_{ij}^{*}=\int_0^1 EI(\xi)\psi_i''(\xi)\psi_j''(\xi)d\xi+\int_0^1 K_1(\xi)\psi_i'(\xi)\psi_j'(\xi)d\xi+\int_0^1 K(\xi)\psi_i(\xi)\psi_j(\xi)d\xi \qquad (20)$$

$$K_{Gij}=\int_0^1 N(\xi)\psi_i'(\xi)\psi_j'(\xi)d\xi \qquad (21)$$

Where K_{ij}^{*} is the first-order elastic stiffness matrix and K_{Gij} is the geometric stiffness matrix, which accounts for the effect of the variable axial force (N) on the bending stiffness of the member.

By applying the principle of virtual displacement along the column element with distributed mass, the mass matrix terms are given by:

$$m_{ij}=\int_0^1 m(\xi)\psi_i(\xi)\psi_j(\xi)d\xi \qquad (22)$$

The structure stiffness and mass matrices can be obtained by assembling each element stiffness and mass matrices according to its nodal displacement. The process of assemblage is described in detail in most stability analysis textbook [1-3]. The critical buckling load and natural frequencies can be derived by solving the eigenvalue problems of following equations. In buckling analysis, we have:

$$(K^{*}+\lambda K_G)\phi=0 \qquad (23)$$

λ are the eigenvalues and ϕ are the related eigenvectors. Under compressive loads, they lead to buckling loads and related eigenmodes.

For the vibration, the eigenvalue problem is put:

$$(K^{*}-\lambda M)\phi=0 \qquad (24)$$

Here λ is related the eigenpulsation of the structure ($\lambda=\omega^2$). ϕ are the related vibration eigenmode.

It is well known that for a system with n DOF, there exist n buckling modes and n vibration modes, but in practice only the lower ones are of interest. A variable iterative algorithm is shown in Fig. 4. It is used for computer applications of the method.

3. Numerical Results

The aim of this section is to investigate the accuracy and efficiency of proposed method. In order to achieve this goal, five comparative examples were made between the natural frequencies and the critical buckling loads of the columns and frames composed of prismatic and non-prismatic members provided by the numerical results of the aforementioned method , available numerical or analytical solutions and finite element method by means of Ansys software [18].

3.1 Example 1

In this example, the stability analysis of three non-prismatic columns, as shown in Fig. 5, with different boundary conditions (fixed-free, hinged-hinged and fixed-hinged) was investigated. Each column had a rectangular cross-section with the depth of the column varying parabolic along its length, and subjected to a compressive axial force P. The modulus of elasticity of the material was assumed $25(GPa)$ and the distribution of moment of inertia $I(\xi)$ is described as follows [6]:

$$I(\xi)=I_A\left(1+C_1(\xi L)^2\right) \qquad (25)$$

Where

$$C_1=\frac{1-DR}{DR.L^2}\ \ and\ \ DR=\frac{d_A}{d_B} \qquad (26)$$

The obtained results have been compared with those obtained by FEM method. To develop a model of non-prismatic column under buckling analysis, the finite element program ANSYS [18] was used.

The abovementioned column has been modeled using BEAM54 of ANSYS software. BEAM54 is a 1D beam element with tension, compression, and bending capabilities. The element has three degrees of freedom at each node: translations in the nodal x and y directions and rotation about the nodal z-axis. This element allows a different unsymmetrical geometry at each end and permits the end nodes to be offset from the centroidal axis of the beam. To obtain almost exact solutions in this model, the column is regarded to be composed of 20 tapered elements.

Table 1 presents the critical axial load parameter λ_{cr} computed by the present method and compared to those obtained by finite element method using Ansys software [18]. This parameter is described as:

$$P_{cr}=\lambda_{cr}\frac{\pi^2 EI_B}{L^2} \qquad (27)$$

According to Table 1, it can be concluded the satisfactory results for engineering requirements can be reached through 3-4 segments and 20 terms of power series by using proposed method.

[DOR: 20.1001.1.23454296.2016.1.2.5.9]
[DOI: 10.29252/nmce.1.2.57]

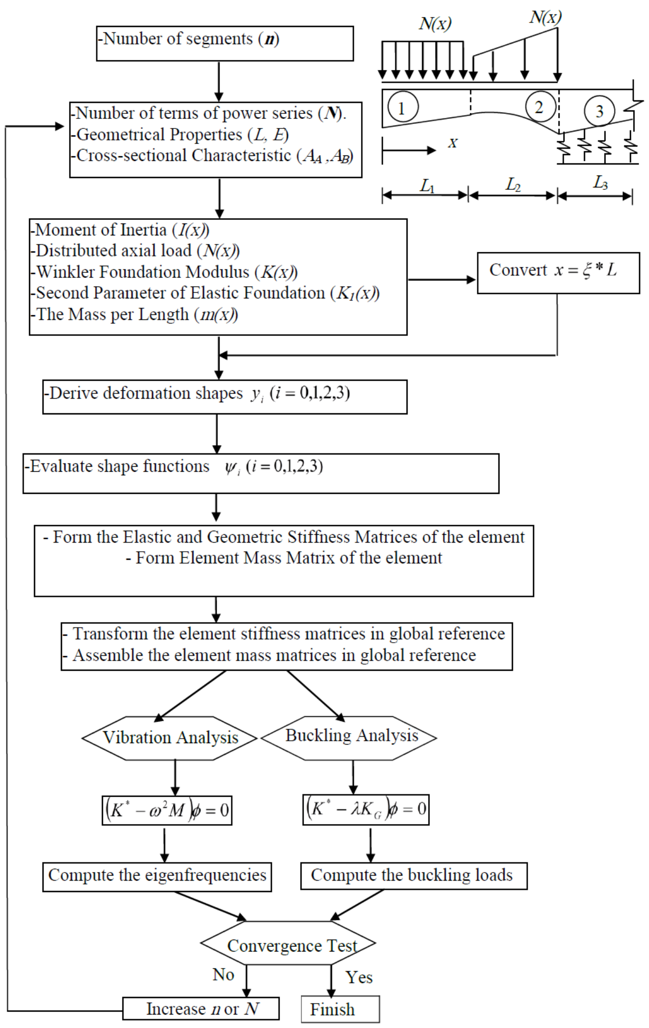


Fig. 4: Iterative algorithms used for free-vibration and buckling analysis of non-prismatic columns on a two-parameter elastic foundation.

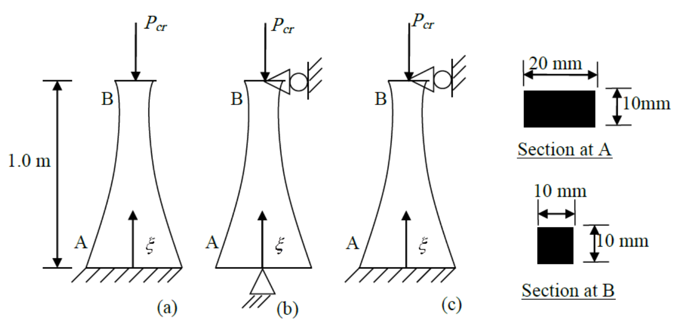


Fig. 5: Non-prismatic columns with different boundary conditions (example 1): (a) Fixed-free; (b) hinged-hinged; (c) fixed-hinged

Table 1: Effect of number of elements (n) and power series number (*N*) on buckling load parameter (λ_{cr}) for non-prismatic column with different boundary conditions. (Example 1)

Figure no.	Present Method				Reference
	Number of Elements, n	Number of terms of power series, *N*			Ansys Software [18]
		10	20	30	
Case a	1	0.468	0.467	0.467	0.463
	2	0.468	0.468	0.468	
	3	0.470	0.470	0.469	
	4	0.467	0.468	0.468	
	5	0.467	0.467	0.467	
Case b	1	2.073	2.079	2.079	1.686
	2	1.757	1.757	1.757	
	3	1.735	1.735	1.735	
	4	1.725	1.725	1.725	
	5	1.719	1.719	1.719	
Case c	1	4.901	5.052	5.055	3.348
	2	3.627	3.625	3.624	
	3	3.474	3.474	3.475	
	4	3.437	3.437	3.437	
	5	3.420	3.420	3.420	

3.2 Example 2

Present method can be applied for the buckling load computation and vibration analysis of prismatic members as well as non-prismatic members. So to check the accuracy and validity of proposed method, three cases involved instability and free vibration analyses of a cantilever column with constants cross section are presented in this example. These cases are on the estimation of Euler buckling load (P_{cr}) (**Case a**), the natural frequency under free-vibration (**Case b**) and the stability analysis (P_{cr}) of a column resting on Winkler type elastic foundation $K=0.4$ (**Case c**).

Fig. 6 provides a graphic depiction of the variation of the relative error with number of segments. After observing the results presented in the Fig.6, the following convulsions can be stated:

(1) There is an excellent agreement between the natural frequency and critical elastic buckling loads obtained by present study and those computed by using Ansys software [18], and exact value by using Euler equation.

(2) The buckling load and natural frequency can be estimated below the acceptable error rate (1%) even by using 4 numbers of segments.

(3) Relative errors (Δ) diminished progressively under 0.1%, as the number of segments increased to 4-10.

(4) It can be concluded not to be required to take more than 5 segments for high accurate solutions involved in stability and free-vibration analyses.

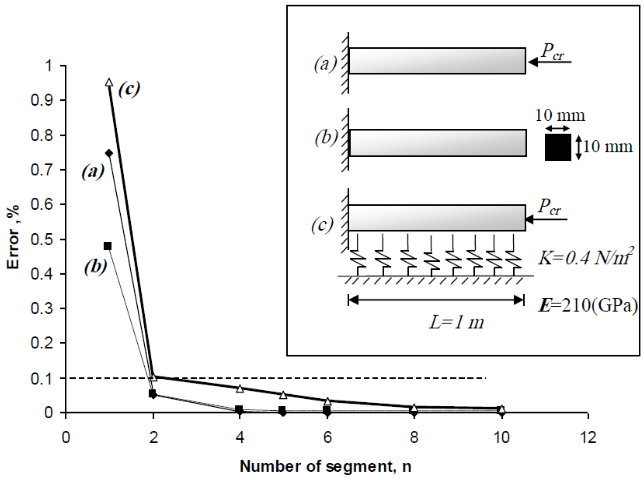


Fig. 6: Relative errors (%) versus the number of segments ***(n)*** variation in the buckling and free-vibration analysis of prismatic column.

3.3 Example 3

The examples in Table 2 are presented to show the accuracy and the exactnesses of presented study to obtain the critical buckling load of different cases.

Case 1 represents a cantilever tapered column resting on Winkler type elastic foundation. The column had a rectangular cross-section with the depth of the column varying linear along its length, and subjected to uniformly distribute axial load W. The length of column ***L***=2m, modulus of elasticity ***E***=25 (GPa), and the distribution of moment of inertia $I(\xi)$ can be written as follows: $I(\xi)=I_A[0.375+0.625*(1-\xi)]$.

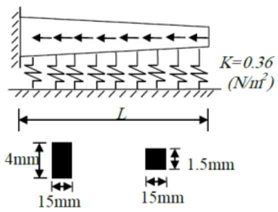
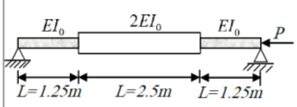
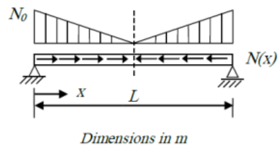
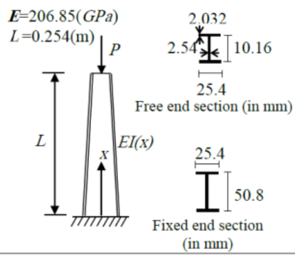
Case 2 shows a stepped simply supported column subjected to axial compression load P. This column composed of three equal parts, with uniform section at each segment, and the central part has a double moment of inertia. Each part length is 1.5m, Young's modulus of elasticity ***E***=210(GPa), and the moment of inertia of the side part is $I_{0=}2.1644e^{-9}$ m^4.

Case 3 gives the exact value of buckling load of a simply supported prismatic column under axial distributed loading of $N(\xi)=N_0(1-2\xi)$ where $N_0=1$ is the value at the point $\xi=0$ and $\xi=1$.

Case 4 deals with the linear buckling of web- tapered cantilevers I section, the web height is made to vary linearly along the length so that, at the free end of cantilever column, the height is diminished. The material and geometrical properties is as shown in the figure.

On the basis of these comparative results presented in the Table 2, it can be stated that, there is an excellent agreement between the critical buckling loads obtained by present method using 20 terms of power series expansions and the other available analytical or numerical results.

Table 2: Comparison of the present analysis results with the other results on the buckling loads of different columns. (Example 3)

	Number of segment	Proposed Method	Reference
Case1 	3 4	237.07 226.61	225.61 Ansys solution [18]
Case 2 	3-3-3	300.67	300.67 (N) Ansys solution [18]
Case 3 	5	$2.06\pi^2\frac{EI}{L^2}$	$2.065\pi^2\frac{EI}{L^2}$ Girgin [14] $2.064\pi^2\frac{EI}{L^2}$ Timoshenko [2]
Case 4 	6	242.1 (N)	241.08 (N) Karabalis [7]

3.4 Example 4

Table 3 provides some information on the convergence of first frequency computed by the present solution with those obtained by using Ansys software [18] and other available numerical and analytical methods. **Case 1** presents the value of natural frequency of a uniform member resting on two parameter elastic foundation for *L=1m, EI=1(Nm²), N=10N, K=0.4,* K_1 *=1.1.*

Case 2 denotes a uniform beam resting on variable Winkler elastic foundation $K(x)=1000(1-0.4\xi^2)$ with *L=1m, EI=1(Nm²)* and $\rho A=1(kg/m)$.

Case 3 gives the value of natural frequency of a tapered beam composed of two elements. The variable functions are:

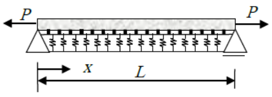
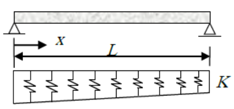
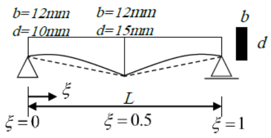
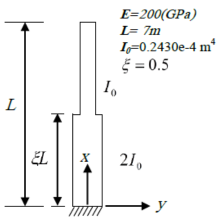
$$A(\xi)=A_0(1+\xi) \quad ,I(\xi)=I_0(1+\xi)^3 \text{ for } \quad 0\le\xi\le 0.5;$$
$$A(\xi)=A_0(2-\xi) \quad ,I(\xi)=I_0(2-\xi)^3 \text{ for } \quad 0.5\le\xi\le 1,$$

which L=1m, $E = 0.3$ GPa and $\rho = 10000$ kg/m^3.

Case 4 signifies the fundamental natural frequency of a steel stepped cantilever column of length ***L***. This column has an abrupt change in cross-section of length $\xi = 0.5$. The geometrical and material properties of column are shown in its figure.

As it is observed in Table 3, there is a good agreement between the results of proposed method and the results of other numerical and theoretical methods in terms of natural frequency of prismatic and non-prismatic members. This example prompts again the efficiency of the current method.

Table 3: The circular frequency $\omega(rad/s)$ for column with different boundary conditions. (Example 4)

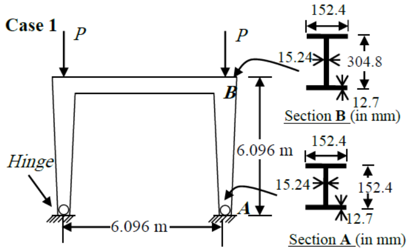
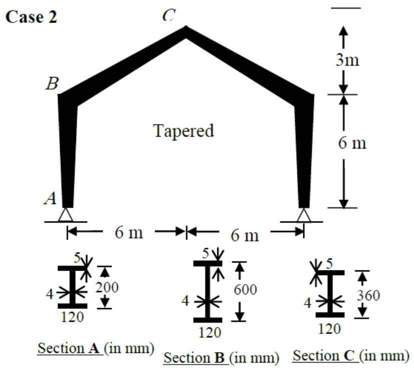
	Number of segment	Present Method	Reference
Case 1 	2 5	14.004 14.001	14.12 Girgin [14] 14.183 Girgin [14]
Case 2 	3 5	31.567 31.34	31.331 Girgin [14] 31.317 Girgin [14]
Case 3 	2-2 5-5	6.515 6.555	6.553 Girgin [14]
Case 4 	4-4	5.933	5.932 Ansys [18]

3.5 Example 5

The last example is an investigation of stability analysis of steel frames composed of tapered and uniform members**.** This example presents the accuracy of proposed study to calculate the buckling load of plane frames **Case 1** considers the steel portal frame of I-cross section with geometry and material properties as shown in figure**. Case 2** presents the steel gable frame having ***I*** -section and the *AB* and *BC* are tapered members which the total depth of cross-section varies linearly along its length. The material properties of member and geometry are indicated in the figure

On the basis of these comparative results presented in the Table 4 can be stated that, there is an excellent agreement between the critical buckling loads obtained by present method and other available analytical or numerical methods.

Table 4: Comparison of the present analysis results with the other results on the buckling loads of different steel frames.(Example 5)

	Present Study	Reference
Case 1 	792.3 (kN)	792 (kN) Karabalis [6]
Case 2 	84.7 (kN)	85.00 Saffari [7]

4. Conclusion

In this paper, a numerical procedure based on power series expansions and the principle of virtual work is used to derive stiffness and mass matrices of non-prismatic column resting on variable two parameter elastic foundations. Power series approach is used to solve the fourth order differential equation with variable coefficients and determine the four sets shape functions of non-uniform members. In turn, based on the principle of internal virtual work along the element axis, the element matrices for the buckling and free-vibration analysis of non-prismatic beam columns were obtained.

The efficiency and accuracy of this method were provided by several comparable numerical examples. The proposed method can be applied in various form of non-prismatic member under axially concentrated loads; fully resting on variable one- or two-parameter elastic foundations; variable mass per unit length. Furthermore; it can be used to evaluate both natural frequency and buckling load concurrently. As demonstrated in the numerical examples section, the results obtained using the proposed computations are in close agreement to those obtained by the rigorous analysis. In most cases, the natural frequencies and critical buckling loads of non-uniform members subjected to several effects can be generated with very good accuracy for engineering problems, within an error of 0.01%–0.3%, by considering only few elements. The efficiency of the method is then confirmed.

Appendix A

With the aim of the symbolic software MATLAB [19], $y_i(i=0,1,2,3)$ are derived. The first few terms are expressed below:

$$y_0(\xi)=1-\frac{\xi^4}{24EI_0^*}\left[L^4\left(K_0^*\right)\right]-\frac{\xi^5}{120EI_0^*}\left[-3\frac{L^4I_1^*}{I_0^*}\left(K_0^*\right)+L^4\left(K_1^*\right)\right]$$
$$-\frac{\xi^6}{360EI_0^*}\left[-\frac{2I_1^*}{I_0^*}\left(-3\frac{L^4I_1^*}{I_0^*}\left(K_0^*\right)+L^4\left(K_1^*\right)\right)+L^4\left(K_2\right)\right.$$
$$\left.-\frac{(12EI_2^*+L^2\left(N_0^*-K_{1_0}^*\right))}{2EI_0^*}\left(L^4\left(K_0^*\right)\right)\right]+\ldots$$

$$y_1(\varepsilon)=\xi-\frac{\xi^4}{24EI_0^*}\left[L^2(N_1^*-K_{11}^*)\right]$$
$$-\frac{\xi^5}{120EI_0^*}\left[L^2\left(-3\frac{I_1^*}{I_0^*}(N_1^*-K_{11}^*)+2(N_2^*-K_{12}^*)\right)+L^4\left(K_0^*\right)\right]$$
$$-\frac{\xi^6}{360EI_0^*}\left[-\frac{2I_1^*}{I_0^*}\left(L^2\left(-3\frac{I_1^*}{I_0^*}(N_1^*-K_{11}^*)+2(N_2^*-K_{12}^*)\right)+L^4\left(K_0^*\right)\right)\right.$$
$$+L^2\left(3(N_3^*-K_{13}^*)+L^2K_1^*\right)$$
$$\left.-\frac{(12EI_2^*+L^2(N_0^*-K_{10}^*))}{2EI_0^*}(L^2(N_1^*-K_{11}^*))\right]+\ldots$$

$$y_2(\varepsilon)=\xi^2-\frac{\xi^4}{24EI_0^*}\left[4EI_2^*+2L^2(N_0^*-K_{10}^*)\right]$$
$$-\frac{\xi^5}{120EI_0^*}\left[-3\frac{I_1^*}{I_0^*}\left(4EI_0^*+2L^2(N_0^*-K_{10}^*)\right)+12EI_3^*+4L^2(N_1^*-K_{11}^*)\right]$$
$$-\frac{\xi^6}{360EI_0^*}\left[-\frac{2I_1^*}{I_0^*}\left(-3\frac{I_1^*}{I_0^*}\left(4EI_0^*+2L^2(N_0^*-K_{10}^*)\right)+12EI_3^*+4L^2(N_1^*-K_{11}^*)\right)\right.$$
$$+24EI_4^*-\frac{(12EI_2^*-L^2(N_0^*-K_{10}^*))}{2EI_0^*}\left(4EI_2^*-2L^2(N_0^*-K_{10}^*)\right)$$
$$\left.-L^2\left(6(N_2^*-K_{12}^*)+K_0^*\right)\right]+\ldots$$

$$y_3(\varepsilon)=\xi^3-\frac{\xi^4}{24EI_0^*}\left[12EI_1^*\right]$$
$$-\frac{\xi^5}{120EI_0^*}\left[-\frac{I_1^*}{I_0^*}\left(36EI_1^*\right)+36EI_2^*+6L^2(N_0^*-K_{10}^*)\right]$$
$$-\frac{\xi^6}{360EI_0^*}\left[-\frac{2I_1^*}{I_0^*}\left(-\frac{I_1^*}{I_0^*}\left(36EI_1^*\right)+36EI_2^*+6L^2(N_0^*-K_{10}^*)\right)+72EI_3^*\right.$$
$$\left.+9L^2(N_1^*-K_{11}^*)-\frac{(12EI_2^*+L^2(N_0^*-K_{10}^*))}{2EI_0^*}(12EI_1^*)\right]+\ldots$$

References

[1] Chen WF, Lui EM. Structural stability, theory and implementation. New York, Elsevier, 1987.

[2] Timoshenko SP, Gere JM. Theory of elastic stability. 2nd Ed. New York, McGraw-Hill, 1961.

[3] Bazant ZP, Cedolin L. Stability of structures. Elastic, Inelastic, Fracture and Damage Theories. New York, Dover Publications, 1991.

[4] Banerjee JR, Williams FW. Exact Bernoulli–Euler dynamic stiffness matrix for a range of tapered beams. Int J Numer Methods Engng 1985; 21:2289–302

[5] Frieman Z, Kosmatka JB. Exact stiffness matrix of a non-uniform beam-extension, torsion and bending of a Bernoulli–Euler beam. Comput Struct 1992; 42(5):671–82.

[6] AL-Sadder S.Z. Exact expression for stability functions of a general non- prismatic beam – column member. Journal of Constructers of Steel Research 2004; (60): 1561-84.

[7] Karabalis DL. Static dynamic and stability analysis of structures composed of tapered beams. J Comput Struct 1983; 16(6):731–48.

[8] Rahai A.R, kazemi S. Buckling analysis of non-prismatic columns based on modified vibration modes. Communication in Non linear Science And Numerical Simulation 2008; (13) 1721-35.

[9] Jategaonkar R, Chehil DS. Natural frequencies of a beam with varying section properties. J Sound Vibration 1989; 133:303–22.

[10] Saffari H, Rahgozar R, Jahanshahi R. An efficient method for computation of effective length factor of columns in a steel gabled frame with tapered members. Journal of Constructers of Steel Research 2008; (64) 400-406.

[11] Ayvaz Y. Application of modified Valsov model to free vibration analysis of beam resting on elastic foundation. Journal of Sound and vibration 2002; (255) 111-127.

[12] Eisenberger M, Clastornik J. Beams on variable two-parameter elastic foundation. J Engng Mech ASCE 1987; 113(10):1454–66.

[13] Matsunaga H. Vibration and buckling of deep beam–columns on two parameter elastic foundations. J Sound Vibration 1999; 228(2): 359–76.

[14] Girgin Z.C, Girgin K. A numerical method for static or dynamic stiffness matrix of non-uniform members resting on variable elastic foundations. Engineering Structures 2005; 27:1373–1384.

[15] Avramidis I.E, Mofidis K. Bending of beams on three-parameter elastic foundation. International Journal of Solids and Structures 2006; (43) 357-375.

[16] Asgarian B, Soltani M, Mohri F. Lateral-torsional buckling of tapered thin-walled beams with arbitrary cross-sections. Thin-Walled Structures 2013; 62: 96-108.
[17] Soltani M, Asgarian B, Mohri F. Elastic instability and free vibration analyses of tapered thin-walled beams by the power series method. Journal of Constructional Steel Research 2014, 96: 106-126.
[18] ANSYS, Version 5.4, Swanson Analysis System, Inc, 2007
[19] MATLAB Version 7.6. MathWorks Inc., USA, 2008.

[DOR: 20.1001.1.23454296.2016.1.2.5.9]
[DOI: 10.29252/nmce.1.2.57]